Optimization

We now move on to one of the most important kinds of applications of calculus—Optimization. *Better, faster, stronger; we have the technology!* To *optimize* something means to make it as good as it can be. In many cases, this can be formulated in terms of maximizing or minimizing something—that is, finding where maximums and minimums occur. Hey! Derivatives can help us with that!

So let's say you make a function that represents the profits you make by selling a product, then of course you'd want the best way to maximize this function (and hence your profits). Or if a function represents your costs, you'd want to minimize that one. You can have a function that describes the amount of material that is used to build something. What's the best way to build it to fulfill its purpose, but use the minimum amount of material possible?—because save the whales and help the Earth and all that. Of course, you'd also like to minimize the cost of spending on materials (because it always comes back to money…).

Another dimension to this kind of problem is the idea of *constraints*. We usually don't have unlimited resources to get something done. So optimizing something within constraints is also important (the amount of material we can use is often limited, or the amount of money we can spend, etc.). Calculus helps us with all these considerations.[1] And the star is, once again, the derivative.

Like *Related Rates*, there is a sequence of steps that can be used to solve virtually any optimization problem you'd be faced with (in this course).

The steps (to optimize something):

1. **Read the problem carefully!** Read the problem carefully. The goal is to understand the problem well enough to do step 2 (and 3), as well as recognize it as an optimization problem.
2. **If possible/necessary, draw and label a diagram** which describes the problem. The geometry of the situation will often give you ideas of what equations you want to set up for the next step.
3. **Set up your constraint and objective equations.** These equations will be based on the information given (and sometimes the diagram). The *constraint equation is an equation that describes your constraints or restrictions* in terms of the variables you set up. *The objective equation describes the quantity that you want to maximize or minimize*, i.e. your objective.
4. **Solve for one of the variables in your constraint equation and plug it into your objective equation.** Your objective will often have more variables than you can handle with single variable calculus. Using your constraint equation(s) to solve for one or more variables in terms of the others, allows us to plug these expressions into our objective and write it as a single variable function. We can then use calculus as we've learned. This step can become a lot harder with multivariable calculus. Thank heavens for calculus 3.
5. **Find the (absolute) maximum or minimum of this new objective function.** Whether you should find the max or min depends on the problem. Sometimes you'll want to maximize (like profits) sometimes you want to minimize (like cost), keep in mind what it is your objective to do: make something as large as possible or as small as possible. Of course, this step requires finding the derivative and its critical points for the objective function. In practice, it is often the local maximums or minimums that will solve your problem, and the endpoints will rarely give the solution—but consider them anyway!
6. **Answer the problem!** Give the people what they want. Pay attention to what you are asked to find and give the required answer!

And that's it. Following the steps above will help you solve pretty much any optimization problem, certainly any one that you'd be given in this class.

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[1] There are cases in which calculus is not the best way to approach an optimization problem, but such cases are outside the scope of our class—and any undergraduate calculus class.

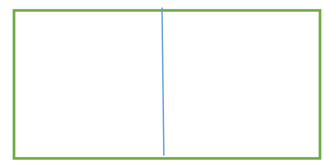
(FYI: I would hope this is obvious by now, but I will mention this just in case. On all the handouts that I've given that shows steps, the words in bold describe the memory cues I want you to assimilate. Read through the steps carefully the first few times you do the problems, but eventually, you want to get to the point where remembering only the things written in bold is enough for you to solve a problem.)

We will illustrate the above steps with an "easy" problem (problem 1), then move on to problems more like what you'd see on an exam or final.

1. A farmer has 100 feet of fencing and he wishes to make a rectangular enclosure. If he wants the enclosure to encompass the maximum possible area so that he can plant the maximum number of crops, what should the dimensions of the enclosure be?

2. Design an open rectangular box with square ends (on left and right side), having a volume of 36 cubic inches, that minimizes the amount of materials required for construction. Save the Earth!

3. A rectangular corral of 162 square meters is to be fenced off and then divided by a fence into two sections, as shown in the figure to the right. Find the dimensions of the corral so that the amount of fencing is minimized. Save the whales!



4. A cylindrical can without a top is to be made to hold 8000π cubic meters of liquid. Find the dimensions of the can which will minimize the amount of material needed to make the can. That is, minimize the surface area of the can.

5. The manager of a department store wants to build a 600 square-foot rectangular enclosure in the store's parking lot to display some equipment. Three sides of the enclosure will be built of redwood fencing at a cost of \$14 per running foot. The fourth side will be built of cement blocks, at a cost of \$28 per running foot. Find the dimensions of the enclosure that will minimize the cost of the building materials.

6. (Bonus to try on your own): A storage shed is to be built in the shape of a box with a square base. It is to have a volume of 300 cubic feet. The concrete for the base costs \$8 per square foot, the material for the roof costs \$4 per square foot, and the material for the sides cost \$5 per square foot. Find the dimensions of the most economical shed. How would things change if the last sentence asked: What is the cost to build the most economical shed?

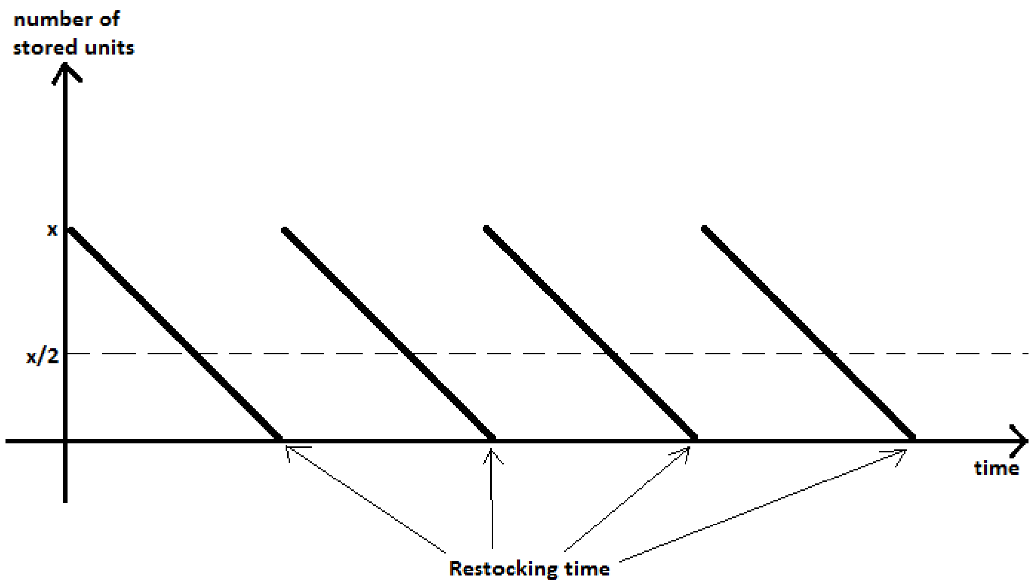
7. (Brownie points bonus. If you can get it by the end of the day that you learn optimization): A travel agency offers a boat tour of several Caribbean islands for 3 days and 2 nights. For a group of 12 people, the cost per person is \$800. For each additional person above the 12-person minimum, the cost per person is reduced by \$20 for each person in the group. The maximum tour group size is 25. What tour group size produces the greatest revenue for the travel agency?

More Optimization – Inventory Cost

Another economical application of optimization is that of inventory cost. That is, the cost of stocking, storing and restocking inventory. The goal here is simple: Minimize the cost associated with storing and ordering inventory while not breaking service.

To make things easier, we shall make some assumptions here:

1. Inventory decreases at a constant rate on average. Hence, the function that describes the number of stored items is a (piece-wise) linear function.
2. Assumption 1 leads to the next assumption, which is that the average number of stored items is $\frac{x}{2}$, where x is the number of items ordered each time. This is illustrated in the graph below.



Under these assumptions, assuming we order x units each time, our objective (over the time interval being considered) becomes: $\boxed{C(x) = \begin{pmatrix} order \\ cost \end{pmatrix}\left(\frac{total\ \#\ of units}{x}\right) + \begin{pmatrix} inventory \\ cost \end{pmatrix}\left(\frac{x}{2}\right)}$

Here, $C(x)$ is the cost of stocking and storing x units—we want to minimize this. The *order cost* refers to all costs associated with ordering x units (delivery costs, labor costs, etc.). *Total # of units* refers to how many total units are sold over some time period, usually a year, hence the $\begin{pmatrix} order \\ cost \end{pmatrix}\left(\frac{total\ \#\ of units}{x}\right)$ term can be thought of as the order cost over the course of the year (or some given time interval). *Inventory cost* refers to the cost of storing the units (rent, labor cost, etc.). And, of course, $\frac{x}{2}$ is the average number of items stored at any given time, hence the $\begin{pmatrix} inventory \\ cost \end{pmatrix}\left(\frac{x}{2}\right)$ term can be thought of as the average storage cost over the time interval being considered, usually a year.

Example: Jhevon sells 1500 hotdogs per year at a steady rate. It costs him \$2 to store each hotdog, and \$10 to order a new shipment of hotdogs. How many times per year should Jhevon order hotdog shipments and how many hotdogs should be in each shipment?