<u>Marginal Functions</u>

An economic application of derivatives is their use as *marginal functions*. That is, functions that describe what occurs at the margins, or outer edges, of production. They do this by helping to answer the question: *given our current level of production, what would happen if we go one step further?* They do this by approximating the change that would occur from producing one more item than has currently been produced. This interpretation can be applied to any of the economic functions (and really, any function in general), so we can discuss *marginal cost*, *marginal revenue*, *marginal profit*, etc.

To see how derivatives can perform this task, we need only look at their definition. We will illustrate using the cost function.

We know that $C'(x) = \lim_{h \to 0} \frac{C(x+h)-C(x)}{h}$. Based on what we know about limits, this is the same as saying $C'(x) \approx \frac{C(x+h)-C(x)}{h}$ when h is small (close to zero). Now, if x is large, say it is the number of items being produced by a large company[1], then $h = 1$ can be considered "small". If we plug in $h = 1$ into the previous equation, we get

$$C'(x) \approx C(x+1) - C(x)$$

But $C(x+1) - C(x)$ is the difference in cost of producing x items and $x+1$ items. This means that the derivative of the cost function at x can be used to approximate the additional cost of producing one more item after x items have been produced.

The same idea works for the other economic functions.

<u>Definition 1:</u> Give a cost function, $C(x)$, or a revenue function, $R(x)$, or a profit function, $P(x)$, we define the following:

- $C'(x)$ is the marginal cost function,
- $R'(x)$ is the marginal revenue function,
- $P'(x)$ is the marginal profit function,

These approximate the addition cost, revenue, profit, respectively, resulting from producing one more item when x items have been produced. That is,

- $C'(x) \approx C(x+1) - C(x)$
- $R'(x) \approx R(x+1) - R(x)$
- $P'(x) \approx P(x+1) - P(x)$

Therefore, the marginal functions tell us:

- *Given that we now produce x items, how much, additionally, would it cost us to produce $x+1$ items (one more item)?*
- *Given our revenue from selling x items, how much additional revenue would we earn if we produced $x+1$ items (one more item)?*
- *Given our current profit margins from producing and selling x items, how much additional profit would we gain by producing $x+1$ items (one more item)?*

---

[1] This means x can be a number in the thousands, tens of thousands, or hundreds of thousands…

You may wonder why derivatives are necessary here. If we care about the difference $C(x+1) - C(x)$, for example, why not just compute both numbers and take the difference? The answer is two-fold: firstly, it is often easier to work with a derivative than to work with the original function twice and take the difference; secondly, if we are at the current level of production, we might not know for sure how the cost structure would change in the future. Using the derivative at the current position is a way to predict what the additional cost would be.

Now let's look at some examples.

Example 1: Suppose the cost of producing x glow-in-the-dark boxer shorts is $C(x) = 2x^2 + 3x$.

(a) What is the additional cost incurred to raise the production level from 10 to 11?

(b) What is the marginal cost when $x = 10$?[2]

Example 2: Suppose the demand function for a product is $p = \frac{25}{\sqrt{x}}$.

(a) Find its marginal revenue function.

(b) What is the marginal revenue when $x = 100$? Interpret this number.

(c) The cost function for the product was found to be $C(x) = x^2 + 2x$, what is the marginal profit function?

(d) What is the marginal profit when $x = 100$, interpret this number.

(e) Is it worth producing one more item?

[2] Note that $C'(10) \approx C(11) - C(10)$. The error here is about 4%, but notice that we're only talking about a few units here. In general, the approximation gets better with a larger number of units.