Exponential Growth and Decay Handout

Any quantity whose rate of change is proportional to its current size is said to undergo *exponential growth* (if it is increasing) or *exponential decay* (if it is decreasing). That is to say the quantity is changing by some fixed percentage amount of its current size over regular intervals. For example, a quantity that is growing at a rate of 2% of its current size per year.

We'll discuss the formulas governing this kind of change, these are summarized below:

<u>The formulas governing Exponential Growth</u>

Equation 1 (The differential equation): $P' = rP$

Equation 2(The initial condition to (1)): $P(0) = P_0$

Equation 3 (The solution to (1)): $P = P_0 e^{rt}$

Equation 4 (Doubling time equation): $t_D = \frac{\ln 2}{r}$

<u>The formulas governing Exponential Decay</u>

Equation 1 (The differential equation): $P' = -rP$

Equation 2(The initial condition to (1)): $P(0) = P_0$

Equation 3 (The solution to (1)): $P = P_0 e^{-rt}$

Equation 4 (Half-life equation): $t_h = \frac{\ln 2}{r}$

In the above formulas:

$P = P(t)$ – the current amount/size at time t,
P' - the rate of change of P,
r – the "growth constant" (resp.) "decay constant", sometimes called the "(relative) growth rate,
P_0 – the initial amount/size (amount at time 0),
t_D – the time taken for P to double in size
t_h - the half-life (the time taken for P to decay to half its initial amount).

<u>Continuous compounding/"Compounded continuously"</u>

This is a term that refers to the continuous compounding of interest on some monetary value (interest is compounded an infinite number of times in some time interval). In this case, P is interpreted as the current balance of monies at time t, while P_0 is the initial principal and r is the interest rate.

Examples for Exponential Growth, Exponential Decay, and Continuous compound interest problems

1. Let $P(t)$ be the current size of the population of a colony of bacteria at time t. At 10am there are 50 bacteria, and at 3pm there are 350. Assume the rate of growth of the population is proportional to its current size.
   (a) Find the relative growth rate of the population,
   (b) Find a formula for $P(t)$,
   (c) What is the size of the population at 4pm?
   (d) What is the rate of growth at 4pm?
   (e) When will the population reach a size of 1500?

2. The half-life of a radioactive substance is 30 years. Suppose we have a 200 mg sample of the substance. Let $P(t)$ be the mass remaining after t years.
   (a) Find the differential equation satisfied by $P(t)$,
   (b) Find a formula for $P(t)$,
   (c) How much will remain after 75 years?
   (d) After how many years will the mass be reduced to 1 mg?
   (e) What is the rate of growth at the time there is 1 mg remaining?

3. A mythical bank pays 5% interest compounded continuously. Suppose you deposit $4000 into an account with this bank.
   (a) Write down a differential equation together with an initial condition whose solution gives the amount in your account at any time t after your initial deposit.
   (b) At what rate (in $/year) is your account increasing when the balance reaches $6000?
   (c) How long will it take for the principal to reach $5000?