1

After learning a core group of basic functions, we will be armed with the tools to create formulas that describe scenarios as diverse as trends in the stock market, world population, historic Olympic wins, the growth of computing power, and the popularity of a new product.

Functions and Models

The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that will be needed in our study of calculus and describe the process of using these functions as mathematical models of real-world phenomena.

1 Functions and Their Representations

Introduction to Functions

Mathematical relationships can be observed in virtually every aspect of our environment and daily lives. Populations, financial markets, the spread of diseases, setting the price of a new product, and the effects of pollution on an ecosystem can all be analyzed using mathematics.

Many mathematical relationships can be considered as *functions*. A function is a correspondence in which one quantity is determined by another. For instance, each day that the US stock market is open corresponds to a closing price of Google stock. We say that the daily closing price of the stock is a function of the date.

For additional illustrations, consider the following four situations.

A. The area A of a square plot of land depends on the length s of one side of the plot. The rule that connects s and A is given by the equation $A = s^2$. With each positive number s there is associated one value of A, and we can say that A is a *function* of s.

B. The human population of the world P depends on the time t. The table gives estimates of the world population for certain years. For instance, when $t = 1950$, $P \approx 2{,}560{,}000{,}000$. But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

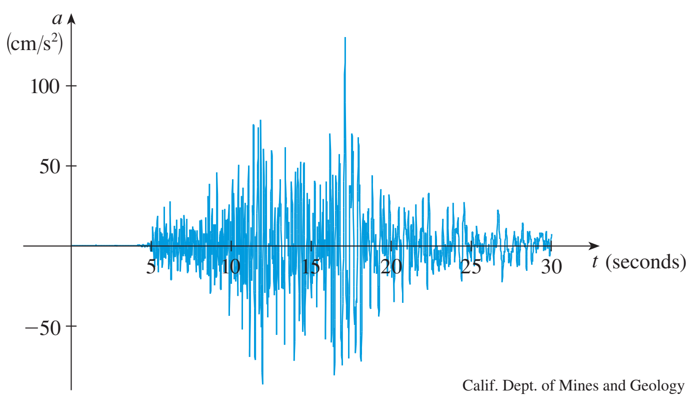


FIGURE 1
rtical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (s, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number. You can think of a function in terms of an input/output relationship, where the function assigns an output value to each input value it accepts.

■ A **function** is a rule that assigns to each input exactly one output.

Notice that while a function can assign only one output to each input, it is perfectly acceptable for two different inputs to share the same output. Although a function can be defined for any sort of input or output, we usually consider functions for which the inputs and outputs are real numbers.

Notation and Terminology for Functions

We typically refer to a function by a single letter such as f. If x represents an input to the function f, the corresponding output is $f(x)$, read "f of x."

The set of all allowable inputs is called the **domain** of the function.

The **range** of f is the set of all possible output values, $f(x)$, as x varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**.

A symbol that represents a number in the *range* of f is called a **dependent variable**.

In Example A, for instance, s is the independent variable and A is the dependent variable. (We can choose the value of s independently, but A *depends* on the value of s.) Using function notation we can write $A = f(s)$, where f represents the area function.

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

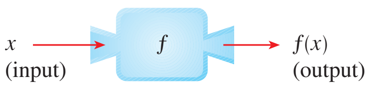


FIGURE 2
Machine diagram for a function f

For example, many cash registers used in retail stores have a button that, when pressed, automatically computes the sales tax to be added to the total. This button can be thought of as a function: An amount of money is entered as an input, and the machine outputs an amount of tax. Both the domain and range of this function are sets of positive numbers that represent amounts of money.

■ EXAMPLE 1 A Price Function

A cafe sells its basic coffee in three different cup sizes: 8, 10, and 14 ounces. They charge \$0.22 per ounce for the drinks.

(a) If the function p is defined so that $p(v)$ is the price of v ounces of coffee, find and interpret the value of $p(10)$.

(b) What are the domain and range of p?

SOLUTION

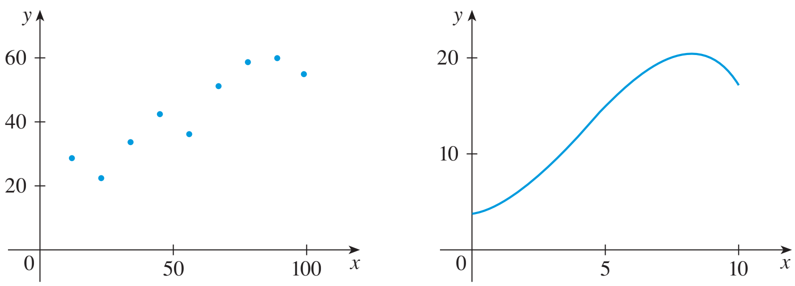
(a) The function value $p(10)$ represents the output (price) of the function when the input is 10 ounces of coffee. Thus $p(10) = \$0.22 \times 10 = \$2.20$.

(b) If we assume that the cafe sells only 8-, 10-, and 14-ounce coffee drinks, then the only allowable inputs to the price function are the three numbers 8, 10, and 14, so the domain of p is the set $\{8, 10, 14\}$. The range is the set of outputs that correspond to the inputs in the domain: $\{1.76, 2.20, 3.08\}$. ■

We use braces { } to list the elements of a set.

Although the rule defining a function may be clear, or you may have a list of inputs and outputs for a function, it is often easiest to analyze a function if you can visualize the relationship between the inputs and outputs. The most common method for visualizing a function is to view its graph. If f is a function, then its **graph** is the set of input-output pairs $(x, f(x))$ plotted as points for all x in the domain of f. In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

If the domain consists of isolated values, as in Example 1, the data are *discrete* and the graph is a collection of individual points, called a *scatter plot*. On the other hand, if the input variable represents a quantity that can vary *continuously* through an interval of values, the graph is a curve or line (see Figure 3). We will define a continuous function more formally in Chapter 2. For now, you can think of a continuous function as one for which you can sketch its graph without lifting your pencil from the paper.



(a) Scatter plot

(b) Continuous function

FIGURE 3
Graphs of functions

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

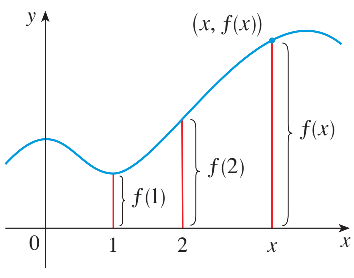


FIGURE 4

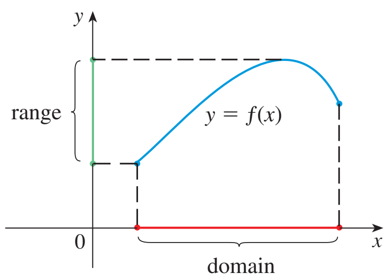
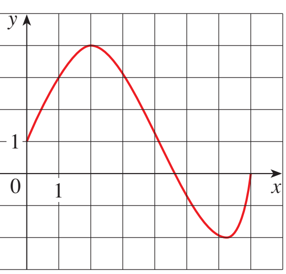


FIGURE 5



URE 6

■ EXAMPLE 2 Reading Information from a Graph

The graph of a function f is shown in Figure 6.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Recall that a closed bracket is used with interval notation to indicate an included value, while an open parenthesis indicates that the endpoint of the interval is not included. For instance, the interval $[2, 5)$ is equivalent to $\{x \mid 2 \leqslant x < 5\}$. An interval is called *closed* if it includes both endpoints; an *open* interval includes neither endpoint. A detailed review is included in Appendix A.

SOLUTION

(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$ ■

Representations of Functions

We have seen four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in several ways, it is often useful to go from one representation to another to gain additional insight into the function. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a square plot of land as a function of its side length is probably the algebraic formula $A(s) = s^2$, though it is possible to compile a table of values or sketch a graph (half a parabola). Because a square has to have a positive side length, the domain is $\{s \mid s > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. For convenience, we can measure $P(t)$ in millions and let $t = 0$ represent the year 1900. Then the table of values of world population at the left provides a convenient representation of this function. If we plot these values, we get the scatter plot in Figure 7.

t	$P(t)$ (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

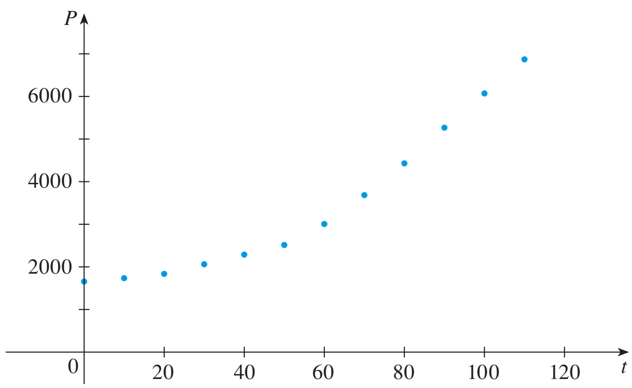


FIGURE 7

This scatter plot is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.5, we obtain the approximation

$$P(t) \approx (1436.53) \cdot (1.01395)^t$$

Figure 8 shows that this function is a reasonably good "fit." Notice that here we have graphed a continuous curve as an approximation to discrete data. We will soon see that the ideas of calculus can be applied to discrete data as well as explicit formulas.

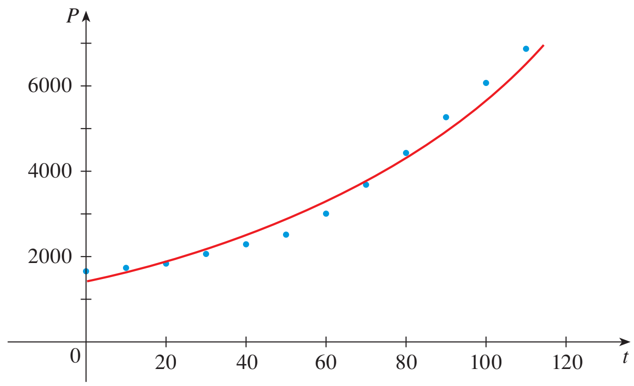


FIGURE 8

C. Again the function is described in words: $C(w)$ is the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2011 is as follows: The cost is 88 cents for up to 1 oz, plus 17 cents for each additional ounce (or less), up to 13 oz. The table of values at the left is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).

unction defined by a table of
es is called a *tabular* function.

(ounces)	$C(w)$ (dollars)
$< w \le 1$	0.88
$< w \le 2$	1.05
$< w \le 3$	1.22
$< w \le 4$	1.39
$< w \le 5$	1.56
⋮	⋮

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

■ EXAMPLE 3 Drawing a Graph from a Verbal Description

When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T

decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 9. ■

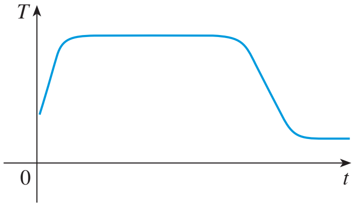


FIGURE 9

A more accurate graph of the function in Example 3 could be obtained by using a thermometer to measure the temperature of the water at 10-second intervals. In general, researchers collect experimental data and use them to sketch the graphs of functions, as the next example illustrates.

t	$N(t)$
1	41.4
3	25.1
5	15.5
7	10.2
9	6.0

■ EXAMPLE 4 A Numerically Defined Function

The data shown in the margin give weekly sales figures for a video game shortly after its release. Let $N(t)$ be the number of copies sold, in thousands, during the week ending t weeks after the game's release. Sketch a scatter plot of these data, and use the scatter plot to draw a continuous approximation to the graph of $N(t)$. Then use the graph to estimate the number of copies sold during the sixth week.

SOLUTION

We plot the five points corresponding to the data from the table in Figure 10. The data points in Figure 10 look quite well behaved, so we simply draw a smooth curve through them by hand as in Figure 11. (Later in this chapter you will see how to find an algebraic formula that approximates the data.)

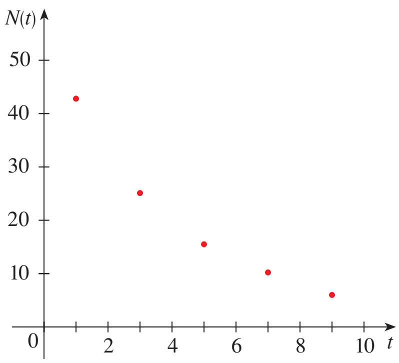


FIGURE 10

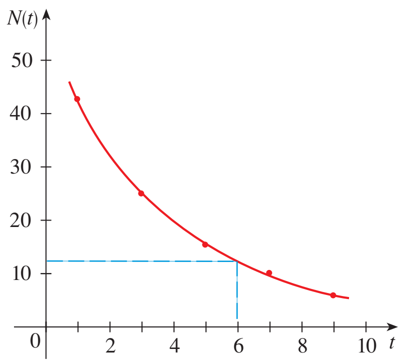


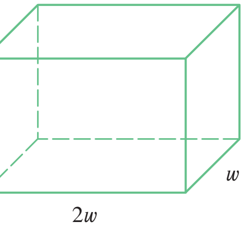
FIGURE 11

From the graph, it appears that $N(6) \approx 12.5$, so we estimate that 12,500 units were sold during the sixth week. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving optimization problems such as maximizing the profit of a company.

■ EXAMPLE 5 Expressing a Cost as a Function

A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.



URE 12

SOLUTION

We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3. Thus

$$\text{volume} = \text{width} \cdot \text{length} \cdot \text{height} = w(2w)h = 10$$

which gives
$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w. ■

In the next two examples we look at functions given by algebraic formulas.

■ EXAMPLE 6 A Function Defined by a Formula

If $f(x) = 2x^2 - 5x + 1$, evaluate

(a) $f(-3)$ **(b)** $f(4) - f(2)$ **(c)** $\dfrac{f(1 + h) - f(1)}{h}$ $(h \neq 0)$

SOLUTION

(a) Replace x by -3 in the expression for $f(x)$:

$$f(-3) = 2(-3)^2 - 5(-3) + 1 = 2 \cdot 9 + 15 + 1 = 18 + 15 + 1 = 34$$

(b) $f(4) - f(2) = [2(4)^2 - 5(4) + 1] - [2(2)^2 - 5(2) + 1] = 13 - (-1) = 14$

(c) We first evaluate $f(1 + h)$ by replacing x by $1 + h$ in the expression for $f(x)$:

$$\begin{aligned} f(1 + h) &= 2(1 + h)^2 - 5(1 + h) + 1 \\ &= 2(1 + 2h + h^2) - 5(1 + h) + 1 \\ &= 2 + 4h + 2h^2 - 5 - 5h + 1 = 2h^2 - h - 2 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\frac{f(1+h)-f(1)}{h} = \frac{(2h^2-h-2)-(2-5+1)}{h}$$

$$= \frac{2h^2-h-2-(-2)}{h}$$

$$= \frac{2h^2-h}{h} = \frac{h(2h-1)}{h} = 2h-1$$ ■

The expression

$$\frac{f(1+h)-f(1)}{h}$$

in Example 6 is called a **difference quotient** and occurs frequently in calculus. We will will begin making use of it in Chapter 2.

■ EXAMPLE 7

Determining the Domain of a Function Defined by a Formula

Find the domain of each function.

(a) $B(r) = \sqrt{r+2}$ **(b)** $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

(a) Because the square root of a negative number is not defined (as a real number), the domain of B consists of all values of r such that $r + 2 \geqslant 0$. This is equivalent to $r \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2-x} = \frac{1}{x(x-1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is $\{x \mid x \neq 0, x \neq 1\}$. ■

The graph of a function is a curve or scatter plot in the xy-plane. But the question arises: Which graphs in the xy-plane represent functions and which do not? This is answered by the following test.

■ **The Vertical Line Test** A curve or scatter plot in the xy-plane is the graph of a function of x if and only if no vertical line intersects the graph more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

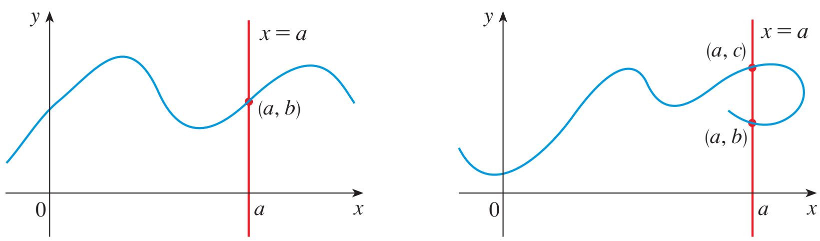


FIGURE 13

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different output values to an input a.

EXAMPLE 8 Using the Vertical Line Test

Determine whether the graph represents a function.

(a)

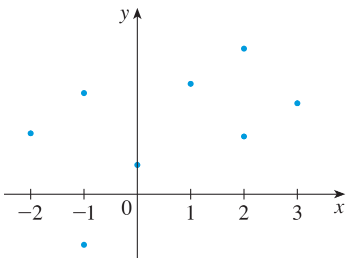


FIGURE 14

(b)

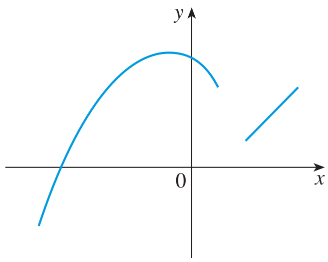


FIGURE 15

SOLUTION

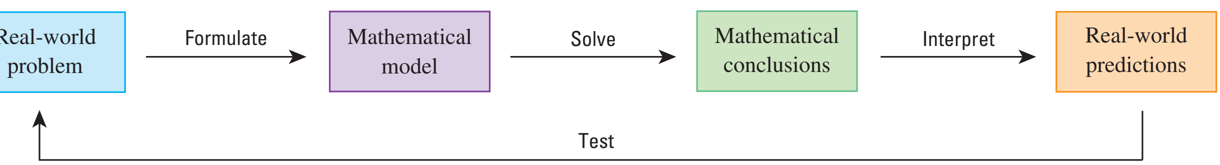
(a) Notice that if we draw a vertical line on the scatter plot in Figure 14 at $x = -1$ or at $x = 2$, the line will intersect two of the points. Therefore the scatter plot does not represent a function.

(b) No matter where we draw a vertical line on the graph in Figure 15, the line will intersect the graph at most once, so this is the graph of a function. Notice that the "gap" in the graph does not pose any trouble; it is acceptable for a vertical line not to intersect the graph at all. ■

Mathematical Modeling

In Example B on page 5, we drew a scatter plot of the world population data and then found an explicit equation that approximated the behavior of the population data. The function P we used is called a *mathematical model* for the population. A **mathematical model** is a mathematical description (usually by means of a function or an equation) of a real-world scenario, such as the demand for a company's product or the life expectancy of a person at birth. Although a function used as a model may not exactly match observed data, it should be a close enough approximation to allow us to understand and analyze the situation, and perhaps to make predictions about future behavior.

Figure 16 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and



URE 16 The modeling process

naming the independent and dependent variables and making assumptions that simplify the situation enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to develop equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from the Internet or a library or by conducting our own experiments) and examine the data in the form of a table or a graph. In the next few sections, we will see a variety of different types of algebraic equations that are often used as mathematical models.

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world situation by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

Keep in mind that a mathematical model is rarely a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature and financial markets have not always been predictable!

Piecewise Defined Functions

In some instances, no single formula adequately describes the behavior of a quantity. A population may exhibit one growth pattern for 20 years but then change to a different trend. In such cases we can use a function with different formulas in different parts of the domain. We call such functions *piecewise defined functions*, and the next two examples illustrate the concept.

EXAMPLE 9 Graphing a Piecewise Defined Function

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(1)$ and sketch the graph.

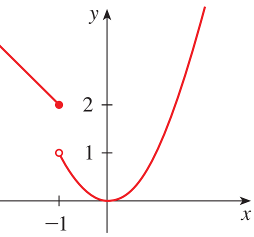
SOLUTION

Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leqslant -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2.

Since $-2 \leqslant -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leqslant -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $1 > -1$, we have $f(1) = 1^2 = 1$.



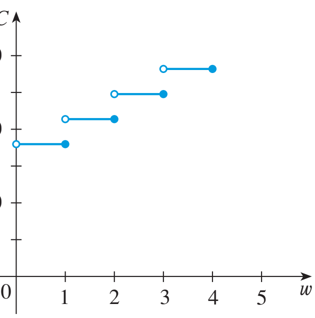
URE 17

How do we draw the graph of f? We observe that if $x \leqslant -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. (Linear equations are reviewed in Section 1.3.) If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 17. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

■ EXAMPLE 10 A Step Function

In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from the table of values on page 6, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \leqslant 1 \\ 1.05 & \text{if } 1 < w \leqslant 2 \\ 1.22 & \text{if } 2 < w \leqslant 3 \\ 1.39 & \text{if } 3 < w \leqslant 4 \\ \vdots & \end{cases}$$



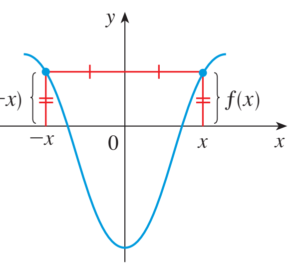
URE 18

The graph is shown in Figure 18. You can see why functions similar to this one are called *step functions*—they jump from one value to the next. ■

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

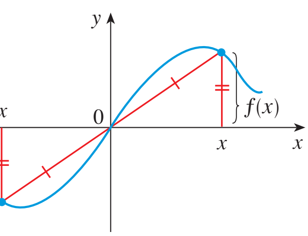


URE 19 An even function

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19). This means that if we have plotted the graph of f for $x \geqslant 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$



URE 20 An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geqslant 0$, we can obtain the entire graph by rotating this portion through 180° about the origin. Note that a function does not have to be either even or odd; many are neither.

EXAMPLE 11 Testing for Symmetry

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

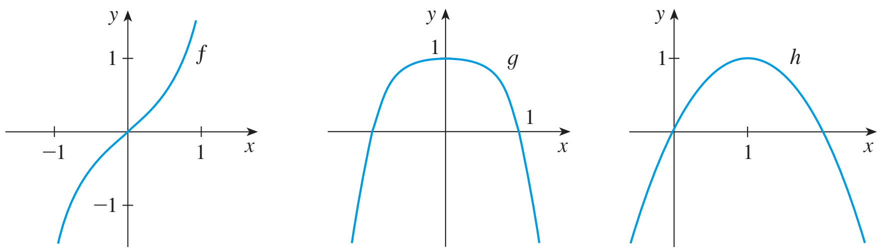


FIGURE 21 (a) Odd function (b) Even function (c) Neither even nor odd

Exercises 1.1

1. **Price function** A nursery sells potting soil for \$0.40 per pound, and the soil is available in 4-lb, 10-lb, and 50-lb bags. If $f(x)$ is the price of a bag of potting soil that weighs x pounds,
   (a) find and interpret the value of $f(10)$.
   (b) determine the domain and range of f.

2. **Price function** An Internet retailer charges \$4.99 to ship an order that totals less than \$25 and \$5.99 for an order up to \$75, and offers free shipping for an order over \$75. If $g(p)$ is the shipping cost for an order totaling p dollars, state the domain and range of g.

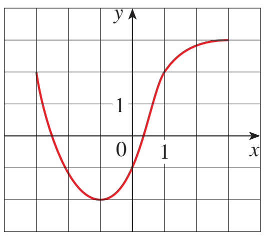
3. **Population function** Let $P(t)$ be the population, in thousands, of a city t years after January 1, 2000. Interpret the equation $P(8) = 64.3$. What does $P(4.5)$ represent?

4. **Blood alcohol content** Let $B(t)$ be the blood alcohol content (measured as the percentage by volume of alcohol in the blood) of a dinner guest t hours after her arrival. Interpret the equation $B(1.25) = 0.06$ in this context.

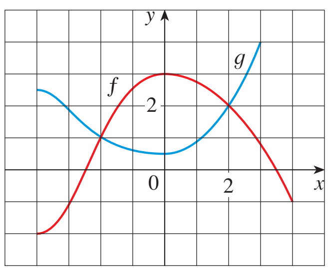
5. **Fuel economy** Let $F(s)$ be the average fuel economy of a particular car, measured in miles per gallon, when the car is being driven at s mi/h. What does the equation $F(65) = 24.7$ say in this context?

Loan payments Let $N(r)$ be the number of \$300 monthly payments required to repay an \$18,000 auto loan when the interest rate is r percent. What does the equation $N(6.5) = 73$ say in this context?

The graph of a function f is given.
(a) State the value of $f(-1)$.
(b) Estimate the value of $f(2)$.
(c) For what values of x is $f(x) = 2$?
(d) Estimate the values of x such that $f(x) = 0$.
(e) State the domain and range of f.



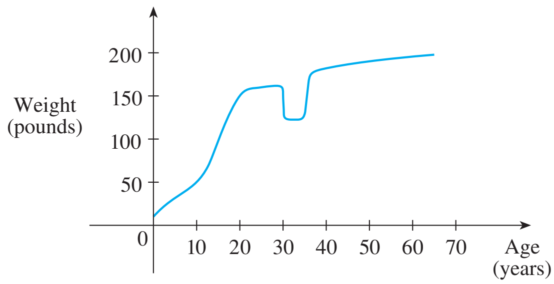
The graphs of f and g are given.
(a) State the values of $f(-4)$ and $g(3)$.
(b) For what values of x is $f(x) = g(x)$?
(c) Estimate the solutions of the equation $f(x) = -1$.
(d) State the domain and range of f.
(e) State the domain and range of g.



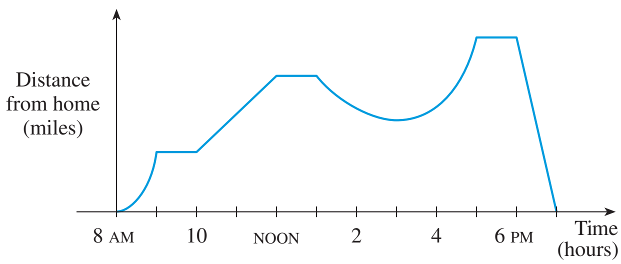
Earthquakes Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

Weight function The graph gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



12. **Distance function** The graph gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



13. **Temperature function** You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

14. **Hours of daylight** Sketch a rough graph of the number of hours of daylight as a function of the time of year.

15. **Temperature function** Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

16. **Market value** Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

17. **Retail sales** Sketch a rough graph of the average daily amount of a particular type of coffee bean (measured in pounds) sold by a store as a function of the price of the beans.

18. **Temperature function** You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

19. **Lawn height** A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

20. Air travel An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.

(a) Sketch a possible graph of $x(t)$.

(b) Sketch a possible graph of $y(t)$.

(c) Sketch a possible graph of the ground speed.

21. Phone subscribers The number N (in millions) of US cellular phone subscribers is shown in the table. (End of year estimates are given.)

t	1996	1998	2000	2002	2004	2006
N	44	69	109	141	182	233

(a) Use the data to sketch a rough graph of N as a function of t.

(b) Use your graph to estimate the number of cell-phone subscribers at the end of 2001 and 2005.

22. Temperature Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Baltimore on September 26, 2007. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	68	65	63	63	65	76	85	91

(a) Use the readings to sketch a rough graph of T as a function of t.

(b) Use your graph to estimate the temperature at 11:00 AM.

23. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a+h)$.

24. If $g(t) = 4t - t^2$, find $g(3)$, $g(-1)$, $g(x)$, $g(x-2)$, and $g(x+h)$.

25–30 ■ Evaluate the difference quotient for the given function. Simplify your answer.

25. $f(x) = x^2 + 1, \quad \dfrac{f(4+h) - f(4)}{h}$

26. $f(x) = 2x^2 - x, \quad \dfrac{f(t+h) - f(t)}{h}$

27. $f(x) = 4 + 3x - x^2, \quad \dfrac{f(3+h) - f(3)}{h}$

28. $f(x) = x^3, \quad \dfrac{f(a+h) - f(a)}{h}$

29. $f(x) = \dfrac{1}{x}, \quad \dfrac{f(x) - f(a)}{x - a}$

30. $f(x) = \dfrac{x+3}{x+1}, \quad \dfrac{f(x) - f(1)}{x - 1}$

31–34 ■ Find the domain of the function.

31. $f(x) = \dfrac{x}{3x - 1}$

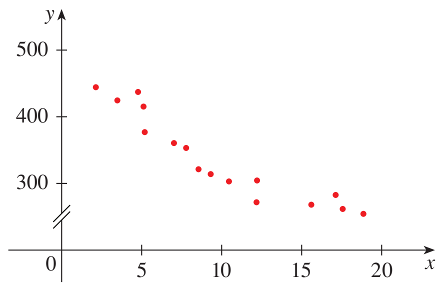
32. $f(x) = \dfrac{3x + 4}{x^2 - x}$

33. $f(t) = \sqrt{2t + 6}$

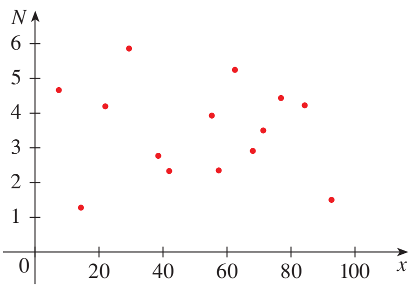
34. $g(u) = \sqrt{u - 4} + 1.5u$

35–36 ■ Determine whether the scatter plot is the graph of a function of x. Explain how you reached your conclusion.

35.

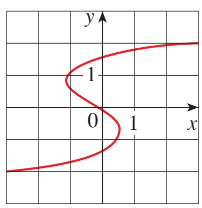


36.

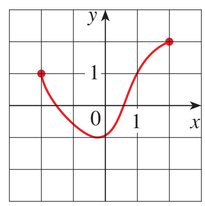


37–40 ■ Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

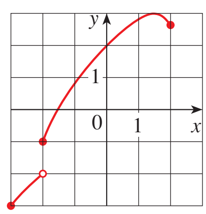
37.



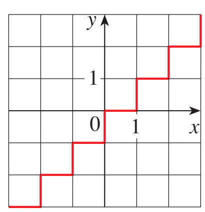
38.



39.



40.



44 ■ Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise
ned function. Then sketch the graph of the function.

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geqslant 0 \end{cases}$$

$$f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geqslant 2 \end{cases}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$$

48 ■ Find a formula for the described function and state
lomain.

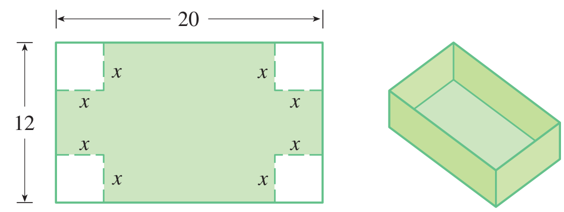
A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

Surface area An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

Height and width A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.

Box design A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



Taxi fares A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C, in dollars, of a ride as a function of the distance x traveled, in miles, for $0 < x < 2$, and sketch the graph of this function.

Income tax In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income beyond \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

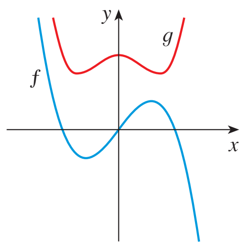
(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

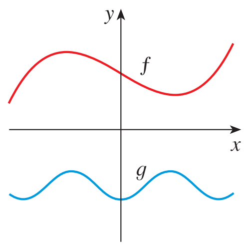
52. The functions in Example 10 and Exercises 50 and 51(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

53–54 ■ Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

53.



54.



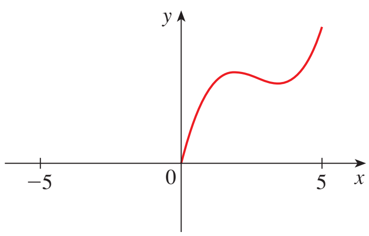
55. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?

(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

56. A function f has domain $[-5, 5]$ and a portion of its graph is shown.

(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.



57–62 ■ Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

57. $f(x) = \dfrac{x}{x^2 + 1}$

58. $f(x) = \dfrac{x^2}{x^4 + 1}$

59. $f(x) = \dfrac{x}{x + 1}$

60. $f(x) = x|x|$

61. $f(x) = 1 + 3x^2 - x^4$

62. $f(x) = 1 + 3x^3 - x^5$

Challenge Yourself

63. If f and g are both even functions and $h(x) = f(x) + g(x)$, is h even? If f and g are both odd functions, is h odd? What if f is even and g is odd? Justify your answers.

64. Window area A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

1.2 Combining and Transforming Functions

In this section we form new functions by combining existing functions in various ways. We also learn how to transform functions by shifting, stretching, or reflecting their graphs. These skills will enable you to use a basic set of functions, studied in the sections ahead, to design specific functions that model a wide variety of applications.

Combinations of Functions

Two functions f and g can be combined to form new functions using the operations of addition, subtraction, multiplication, and division in a manner similar to the way we add, subtract, multiply, and divide real numbers. For instance, we can define a new function h that is the sum of f and g by the equation $h(x) = f(x) + g(x)$. This means that the output of the new function h is the sum of the outputs of the individual functions f and g. This definition makes sense if both $f(x)$ and $g(x)$ are defined. Thus the domain of the function h consists of only those values that belong to both the domain of f and the domain of g.

Suppose a company has two different shipping centers, one on the West Coast and the other on the East Coast. If $W(t)$ is the number of packages shipped from the western facility t weeks after the start of the year, and $E(t)$ is the number of packages shipped from the eastern facility t weeks after the start of the year, then we can define a new function $N(t)$ by

$$N(t) = W(t) + E(t)$$

Thus $N(t)$ measures the combined number of packages sent from both shipping centers t weeks after the start of the year. Notice that the input for each function is the same; if the inputs of two functions are not measuring the same quantities, the sum of the functions is not meaningful.

We can subtract, multiply, or divide functions in a similar way. For instance, $k(x) = f(x)\,g(x)$ means that the output of the function k is the product of the outputs of the functions f and g. The domain of each of these new functions consists of all the numbers that appear in both the domain of f and the domain of g, with the exception that if we divide f by g, we must ensure that no division by 0 will occur. So the domain of $q(x) = f(x)/g(x)$ is all values shared by the domains of f and g where $g(x) \neq 0$.

■ EXAMPLE 1 Combining Two Functions

If $N(v) = \sqrt{v}$ and $T(v) = 3 - v$, find equations and the domains for the functions $A(v) = N(v)\,T(v)$ and $B(v) = N(v)/T(v)$.

SOLUTION

The domain of $N(v) = \sqrt{v}$ is $[0, \infty)$, all the real numbers greater than or equal to 0. The domain of $T(v) = 3 - v$ is $\mathbb{R}$, all real numbers. The domain of $A(v) = N(v)\,T(v)$ consists of those values that are shared by both these domains, namely $[0, \infty)$. The formula for the product function is

$$A(v) = N(v)\,T(v) = \sqrt{v}\,(3 - v)$$

Similarly,

$$B(v) = \frac{N(v)}{T(v)} = \frac{\sqrt{v}}{3 - v}$$

Notice that $T(v) = 0$ when $v = 3$, so 3 must be excluded from the domain of B. Thus the domain of B is all real numbers greater than or equal to 0, except 3. In set-builder notation, we write $\{v \mid v \geqslant 0, v \neq 3\}$. ■

■ EXAMPLE 2 Combining Revenue and Cost Functions

Suppose the annual revenue, in millions of dollars, of a company is $R(t) = 0.2t^2 + 3t + 5$, where t is measured in years and $t = 0$ corresponds to the year 2000. The annual cost, in millions of dollars, for the company is $C(t) = 4t + 9$.

(a) Find a formula for the function $P(t) = R(t) - C(t)$.

(b) Compute and interpret $P(7)$.

SOLUTION

(a)
$$\begin{aligned} P(t) = R(t) - C(t) &= (0.2t^2 + 3t + 5) - (4t + 9) \\ &= 0.2t^2 + 3t + 5 - 4t - 9 \\ &= 0.2t^2 - t - 4 \end{aligned}$$

(b) We can find $P(7)$ by subtracting the output values of the functions R and C, or we can use the formula from part (a) directly:

$$P(7) = 0.2(7^2) - 7 - 4 = -1.2$$

Notice that $P(t)$ is the annual revenue minus the annual cost, so it represents the annual profit for the company. Since $t = 7$ corresponds to 2007, and the output is negative, we know that during 2007 the company lost 1.2 million dollars. ■

Composition of Functions

There is another way of combining two functions to form a new function. As a simple illustration, suppose that a company's annual profit for year t is given by $P(t)$ and the total amount of tax the company pays, $f(P)$, is determined by its profit P. Since the tax paid is a function of profit and profit is, in turn, a function of t, it follows that the amount of tax paid is ultimately a function of t. In effect, the output of the profit function P can be used as the input for the tax function f, and $f(P(t))$ is the amount of tax the company paid during year t. This new function is called the *composition* of the functions P and f.

If we have equations for two functions, we can write a formula for their composition. For example, suppose $y = f(t) = \sqrt{t}$ and $t = g(x) = x^2 + 1$. Now y is a function of t and t is a function of x, so y can be considered as a function of x. We compute this by substitution:

$$y = f(t) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

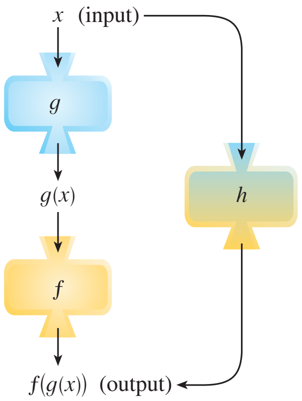


FIGURE 1
The h machine is composed of the g machine (first) and then the f machine.

■ **Definition** Given two functions f and g, the **composition** of f and g is defined by

$$h(x) = f(g(x))$$

The domain of $h(x) = f(g(x))$ is the set of all values x in the domain of g such that $g(x)$ is in the domain of f. In other words, $f(g(x))$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. It is probably easier to picture the composition of f and g with a machine diagram (see Figure 1).

■ EXAMPLE 3 Composing Two Functions

Let $f(x) = x^2$ and $g(x) = x - 3$. If $h(x) = f(g(x))$ and $k(x) = g(f(x))$, compute $h(5)$ and $k(5)$.

SOLUTION

First let's trace the path the input 5 takes under the function h. Since $h(5) = f(g(5))$, we first input 5 into the inner function g, where $g(5) = 2$. The output 2 is then used as an input into the outer function f, which gives an output of $f(2) = 2^2 = 4$. Thus $h(5) = f(g(5)) = f(2) = 4$. Similarly, $k(5) = g(f(5)) = g(25) = 22$. Notice that the original input always goes through the inner function first, and the resulting output is used as an input into the outer function.

We can also write formulas for h and k:

$$h(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$k(x) = g(f(x)) = g(x^2) = x^2 - 3$$

Then it is easy to compute

$$h(5) = (5 - 3)^2 = 2^2 = 4 \qquad \text{and} \qquad k(5) = 5^2 - 3 = 25 - 3 = 22$$ ■

NOTE: You can see from Example 3 that, in general, $f(g(x)) \neq g(f(x))$. Remember, the notation $f(g(x))$ means that the function g is applied first and then f is applied second. In Example 3, $f(g(x))$ is the function that *first* subtracts 3 and *then* squares; $g(f(x))$ is the function that *first* squares and *then* subtracts 3.

■ EXAMPLE 4 Interpreting a Composition of Functions

The altitude of a small airplane t hours after taking off is given by $A(t) = -2.8t^2 + 6.7t$ thousand feet, where $0 \leq t \leq 2$. The air temperature in the area at an altitude of x thousand feet is $f(x) = 68 - 3.5x$ degrees Fahrenheit.

(a) What does the composition $h(t) = f(A(t))$ measure?

(b) Compute $h(1)$ and interpret your result in this context.

(c) Find a formula for $h(t)$.

(d) Does $A(f(x))$ give a meaningful result in this context?

SOLUTION

(a) The hours t that the airplane has been flying is first used as an input into the inner function A, which outputs the altitude of the plane $A(t)$ in thousands of feet. This altitude in turn is used as an input into the outer function f, which outputs a temperature in degrees Fahrenheit. Thus h is the air temperature at the airplane's location t hours after take-off.

(b) The input 1 first enters the function A, giving $A(1) = 3.9$. We then input 3.9 into the function f, which gives $f(3.9) = 54.35$. This means that 1 hour after take-off, the air temperature at the plane's location is 54.35°F.

(c)
$$\begin{aligned} h(t) &= f(A(t)) = f(-2.8t^2 + 6.7t) = 68 - 3.5(-2.8t^2 + 6.7t) \\ &= 9.8t^2 - 23.45t + 68 \end{aligned}$$

Using this direct formula, you can verify that $h(1) = 54.35$ as we found in part (b).

(d) Although we could compute a formula for $A(f(x))$, it wouldn't be a meaningful quantity here. The inner function f outputs a temperature in °F, but this is not an appropriate value to pass to the outer function A as an input, because A is a function of t, a number of hours. ■

So far we have used composition to build complicated functions from simpler ones. But we will see in later chapters that in calculus, it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

■ EXAMPLE 5 Decomposing a Function

If $L(t) = (2t - 1)^3$, find functions f and g such that $L(t) = f(g(t))$.

not important what letter we to represent the variable in the r function f. The function $= x^3$ is the same function as $= a^3$ or $f(q) = q^3$.

SOLUTION

The formula for L says: First double t and subtract 1, then cube the result. One option is to think of $2t - 1$ as the inner function and call it g. Then

$g(t) = 2t - 1$ and $L(t) = (g(t))^3$. The outer function is the cubing function, so if we let $f(x) = x^3$, then

$$L(t) = f(g(t)) = f(2t - 1) = (2t - 1)^3$$

Note that there are other choices we could have made, such as $g(t) = 2t$ and $f(x) = (x - 1)^3$, but the first solution is probably the most useful one. ■

Transformations of Functions

Next we discuss how to modify a function to change the shape or location of its graph. Armed with these techniques, we can use familiar graphs to design functions that will fit a wide variety of applications. The first of these *transformations* we will consider are called **translations**. If you compare the graphs of $y = f(x)$ and $y = f(x) + 3$ in Figure 2, you will notice that the shapes are identical, but the second graph is located 3 units higher on the coordinate plane. The second function increases each output of the first function by 3, so each point on its graph moves 3 units higher. In effect, we have shifted the entire graph upward 3 units. Similarly, $y = f(x) - 3$ shifts the graph of $y = f(x)$ downward 3 units.

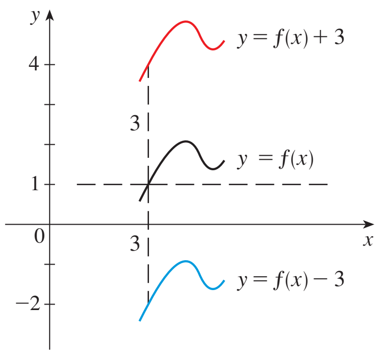


FIGURE 2
Vertical translations of the graph of f

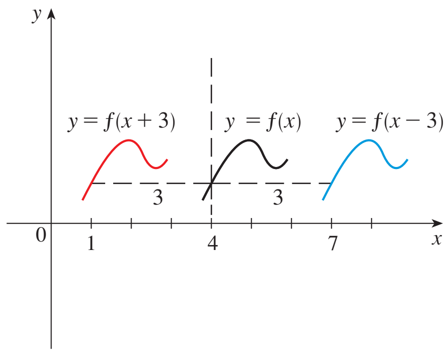


FIGURE 3
Horizontal translations of the graph of f

Next compare the graph of $y = f(x)$ with the graph of $y = f(x + 3)$ in Figure 3. The graph of $y = f(x + 3)$ is the same as the graph of $y = f(x)$ but shifted 3 units to the *left*. To see why this is the case, note that if $g(x) = f(x + 3)$, then $g(1) = f(1 + 3) = f(4)$, so the output corresponding to $x = 4$ in the graph of f is plotted with $x = 1$ in the graph of g, 3 units to the left. Similarly, $y = f(x - 3)$ shifts the graph of f to the right 3 units.

■ **Vertical and Horizontal Shifts** Suppose c is a positive number.

translation of the graph of $y = f(x)$	equation
shift c units upward	$y = f(x) + c$
shift c units downward	$y = f(x) - c$
shift c units to the right	$y = f(x - c)$
shift c units to the left	$y = f(x + c)$

We can also **stretch** (or compress) graphs. For instance, compare the graphs of $y = f(x)$ and $y = 2f(x)$ in Figure 4. The second graph has a shape similar to the first, but it has been stretched vertically by a factor of 2. Each output of the original function is doubled, so the vertical distance between each point of the graph and the x-axis is doubled. If we graph $y = \frac{1}{2}f(x)$, each output is halved, so the graph appears to be compressed vertically (toward the x-axis).

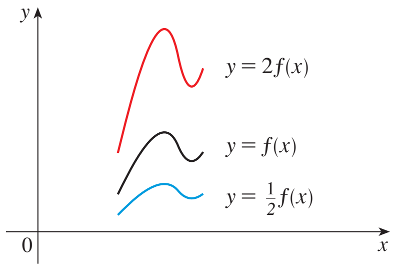


FIGURE 4
Stretching the graph of f vertically

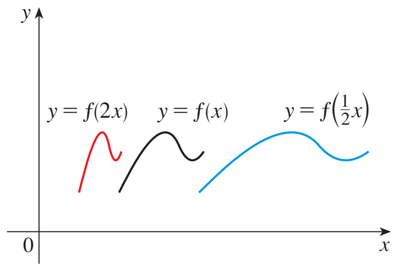


FIGURE 5
Stretching the graph of f horizontally

Now compare the graphs of $y = f(x)$ and $y = f(2x)$ in Figure 5. This time we have compressed the graph horizontally (toward the y-axis) by a factor of 2. To see why this occurs, observe that if $g(x) = f(2x)$, then $g(1) = f(2 \cdot 1) = f(2)$, so the output corresponding to $x = 2$ in the graph of f is plotted with $x = 1$ in the graph of g, half the distance from the y-axis. Similarly, the graph of $y = f\left(\frac{1}{2}x\right)$ is the graph of $y = f(x)$ stretched horizontally by a factor of 2.

■ Vertical and Horizontal Stretching Suppose $c > 1$.

transformation of the graph of $y = f(x)$	equation
stretch vertically by a factor of c	$y = cf(x)$
compress vertically by a factor of c	$y = \frac{1}{c}f(x)$
compress horizontally by a factor of c	$y = f(cx)$
stretch horizontally by a factor of c	$y = f\left(\frac{1}{c}x\right)$

Finally, we can **reflect** graphs in either a vertical or horizontal direction. If we compare the graphs of $y = f(x)$ and $y = -f(x)$ in Figure 6, the graph of $y = -f(x)$ is the graph of $y = f(x)$ but flipped upside down. Each point (x, y) on the original graph is replaced by the point $(x, -y)$, so the graph appears to be reflected about the

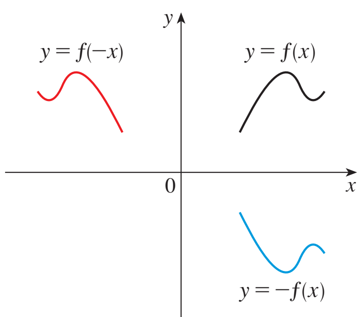


FIGURE 6

x-axis. If you compare the graph of $y = f(x)$ with the graph of $y = f(-x)$ in Figure 6, you'll notice that this time the x-values are made opposite, so the graph appears reflected about the y-axis.

Vertical and Horizontal Reflections

reflection of the graph of $y = f(x)$	equation
reflect about the x-axis	$y = -f(x)$
reflect about the y-axis	$y = f(-x)$

Figure 7 illustrates several combinations of various transformations.

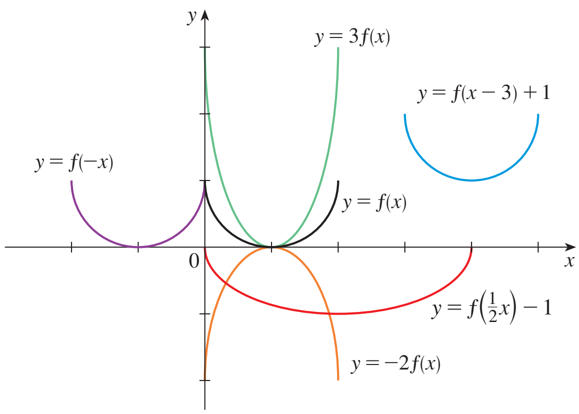


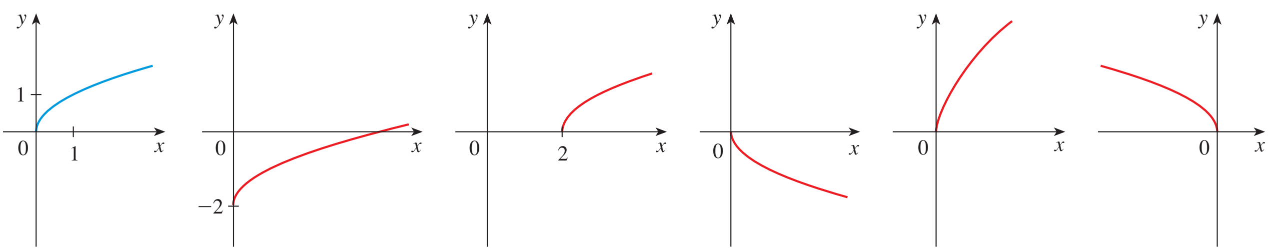
FIGURE 7

EXAMPLE 6 Sketching Transformations of a Function

Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

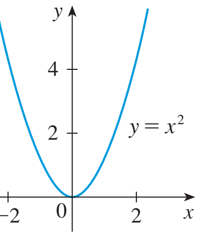
SOLUTION

The graph of the square root function $y = \sqrt{x}$ is shown in Figure 8(a). If we let $f(x) = \sqrt{x}$, then $y = \sqrt{x} - 2 = f(x) - 2$, so the graph is shifted 2 units downward. Similarly, $y = \sqrt{x - 2} = f(x - 2)$ shifts the graph 2 units to the right, $y = -\sqrt{x} = -f(x)$ reflects the graph about the x-axis, $y = 2\sqrt{x} = 2f(x)$ stretches the graph vertically by a factor of 2, and $y = \sqrt{-x} = f(-x)$ reflects the graph about the y-axis.

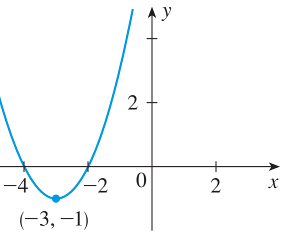


(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x - 2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

FIGURE 8



URE 9



URE 10

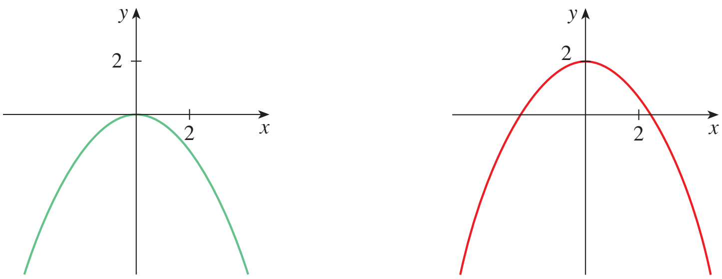
■ EXAMPLE 7 Sketching Multiple Transformations

Given the graph of the function $y = x^2$ shown in Figure 9, sketch the graphs of **(a)** $f(x) = (x + 3)^2 - 1$ and **(b)** $g(x) = -\frac{1}{3}x^2 + 2$.

SOLUTION

(a) The graph of $y = (x + 3)^2$ is the graph of $y = x^2$ shifted 3 units to the left. If we then shift the graph down 1 unit, we have the graph of $f(x) = (x + 3)^2 - 1$ shown in Figure 10.

(b) The graph of $y = -\frac{1}{3}x^2$ is the graph of $y = x^2$ compressed vertically by a factor of 3 and reflected across the x-axis. [See Figure 11(a).] Shift the resulting graph up 2 units to arrive at the graph of $g(x) = -\frac{1}{3}x^2 + 2$ as shown in Figure 11(b).



(a) $y = -\frac{1}{3}x^2$ (b) $y = -\frac{1}{3}x^2 + 2$

FIGURE 11 ■

■ EXAMPLE 8 Interpreting Transformations of Functions

Let $C(x)$ be the amount, in thousands of dollars, that a manufacturer charges for an order of x thousand computer memory chips.

(a) The price (in thousands of dollars) that a rival manufacturer charges to provide x thousand chips is given by $f(x) = C(x) + 12$. How does the rival company's price compare to that of the first company?

(b) What if the amount that the rival company charges for an order of x thousand chips is given by $g(x) = 1.4C(x)$?

(c) What if the rival charges $h(x) = C(x - 2)$ to provide x thousand chips?

SOLUTION

(a) The output of f is always 12 greater than the output of C (for the same input), so the rival supplier charges \$12,000 more for each order than the first manufacturer.

(b) The output of g is 1.4 times the output of C, so the price that the rival company charges is 1.4 times greater, or 40% more, than the first manufacturer's price.

(c) The graph of $h(x)$ is the graph of $C(x)$ shifted two units to the right. This means that for the same price, the rival manufacturer will supply 2000 more chips than the first manufacturer. For instance, if $x = 10$ then $h(10) = C(8)$. ■

Exercises 1.2

1. Class attendance Let $M(t)$ be the number of male students and $F(t)$ the number of female students that attended a math class at a local university on day t of this year. If we define a function g where $g(t) = M(t) + F(t)$, describe what g measures.

2. Price of gas Let $A(x)$ be the total amount charged to a consumer for x gallons of premium gasoline at a particular gas station, and let $T(x)$ be the total amount of tax the station pays for x gallons of the gasoline. What does the function $f(x) = A(x) - T(x)$ measure?

3. Bank holdings Let $g(n)$ be the amount of gold, in ounces, that a bank has in its vault at the end of the nth day of this year, and let $v(n)$ be the value, in dollars, of one ounce of gold at the end of the nth day of this year. What does the function $f(n) = g(n)v(n)$ measure?

4. Investments Let $P(t)$ be the daily closing price of one share of General Electric stock t days after January 1, 2010, and let $Q(t)$ be the number of shares owned by a pension fund at the end of that same day. What does the function $g(t) = P(t)Q(t)$ measure?

5. Crops A farm devotes $A(x)$ acres of its land to growing corn during year x. If $B(x)$ is the number of bushels of corn the farm yielded during year x, what does the function $C(x) = B(x)/A(x)$ represent?

6. Phone usage Let $M(n)$ be the total number of minutes Kathi talked on her cellular phone during the nth month of last year, and let $C(n)$ be the amount she paid for her phone service during that month. What does the function $h(n) = C(n)/M(n)$ represent?

7. Salary An employee's annual salary, in thousands of dollars, is given by $S(t) = 42 + 1.8t$, where t is the year with $t = 0$ corresponding to 2000, and $C(t) = 16.4 + 0.6t$ is the total amount of commissions, in thousands of dollars, the employee earned that year.

(a) Find a formula for the function $f(t) = S(t) + C(t)$.

(b) Compute $f(4)$ and interpret your result in this context.

8. Revenue and profit The annual revenue of a small store, in thousands of dollars, is given by $R(t) = 645 + 21t$, where t is the year, with $t = 0$ corresponding to 2000. Similarly, the store's annual profit is given by $P(t) = 175 + 16t - 0.3t^2$.

(a) Write a formula for the annual cost function $C(t)$ for the store.

(b) Compute $C(3)$ and interpret your result in this context.

9. If $f(x) = x^2 - 5x$ and $g(x) = 3x + 12$, write a formula for each of the following functions.

(a) $A(x) = f(x) + g(x)$ **(b)** $B(x) = f(x) - g(x)$

(c) $C(x) = f(x)g(x)$ **(d)** $D(x) = f(x)/g(x)$

10. If $p(x) = \sqrt{x + 1}$ and $q(x) = 2x - 4$, write a formula for each of the following functions. What is the domain?

(a) $A(x) = p(x) + q(x)$ **(b)** $B(x) = p(x) - q(x)$

(c) $C(x) = p(x)q(x)$ **(d)** $D(x) = p(x)/q(x)$

11. If $f(x) = x^2 + 1$, $g(t) = 4t - 2$, $A(t) = f(g(t))$, and $B(x) = g(f(x))$, compute $A(3)$ and $B(3)$.

12. If $h(n) = 2 - 5n$, $p(n) = n^2 - 3$, $u(n) = h(p(n))$, and $v(n) = p(h(n))$, compute $u(2)$ and $v(2)$.

13. If $M(t) = t + \sqrt{t}$, $N(t) = 3t + 7$, $C(t) = M(N(t))$, and $D(t) = N(M(t))$, compute $C(3)$ and $D(4)$.

14. If $f(t) = t^3 + 2$, $g(x) = 2x + 3$, $p(x) = f(g(x))$, and $r(t) = g(f(t))$, compute $p(-1)$ and $r(-2)$.

15–20 ■ Find the functions $p(x) = f(g(x))$ and $q(x) = g(f(x))$.

15. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

16. $f(x) = 1 - x^3$, $g(x) = 1/x$

17. $f(x) = x^3 + 2x$, $g(x) = 1 - \sqrt{x}$

18. $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$

19. $f(x) = x + \dfrac{1}{x}$, $g(x) = x + 2$

20. $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$

21. Surfboard production Let $N(t)$ be the number of surfboards a manufacturer produces during year t. If $P(x)$ is the profit, in thousands of dollars, the manufacturer earns by selling x surfboards, what does the function $f(t) = P(N(t))$ represent?

Car maintenance If $C(m)$ is the average annual cost for maintaining a Honda Civic that has been driven m thousand miles and $f(t)$ is the number of miles on Sean's Honda Civic t years after he purchased it, what does the function $g(t) = C(f(t))$ represent?

Carpooling As fuel prices increase, more drivers carpool. The function $f(p)$ gives the average percentage of commuters who carpool when the cost of gasoline is p dollars per gallon. If $g(t)$ is the average monthly price (per gallon) of gasoline, where t is the time in months beginning January 1, 2011, which composition gives a meaningful result, $f(g(t))$ or $g(f(p))$? Describe what the resulting function measures.

Home prices People are moving into a small community and driving the home prices higher. Suppose $p(t)$ is the population of the community t years after January 1, 2000, and $f(n)$ is the median home price when the population of the area is n people. Which function gives a meaningful result, $p(f(n))$ or $f(p(t))$? What does it represent in this context?

Scuba diving The pressure a scuba diver experiences at a depth of d feet is approximately $P(d) = 14.7 + 0.433d$ PSI (pounds per square inch). Suppose that for the first portion of Paul's dive, his depth after m minutes is $f(m) = 0.5m + 3\sqrt{m}$ feet.

(a) Write a formula for the function $A(m) = P(f(m))$. What does A measure?

(b) Compute $A(25)$ and interpret your result in this context.

Electric power A town produces a portion of its electricity using windmills. Suppose that with winds that average s mi/h, the windmills generate $p(s) = \sqrt{1400s}$ kilowatts of power. The town estimates that $f(x) = 0.34x$ is the number of people that can be supported by a power level of x kilowatts.

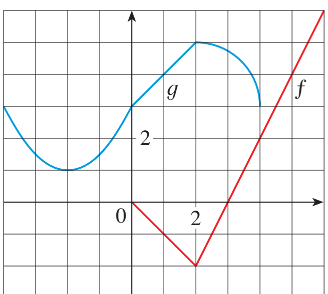
(a) Write a formula for the function $r(s) = f(p(s))$. What does r measure?

(b) Compute $r(18)$ and interpret your result in this context.

Use the given graphs of f and g to evaluate each expression.

(a) $f(g(2))$ **(b)** $g(f(0))$

(c) $f(g(0))$ **(d)** $f(f(4))$



28. Use the table to evaluate each expression.

(a) $f(g(1))$ **(b)** $g(f(1))$

(c) $g(f(3))$ **(d)** $f(g(6))$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

29–32 ▪ Find functions f and g so that $h(x) = f(g(x))$.

29. $h(x) = (x^2 + 1)^{10}$

30. $h(x) = \sqrt{x^3 - 1}$

31. $h(x) = \sqrt{2x^2 + 5}$

32. $h(x) = \dfrac{1}{x^2 - 5}$

---

33. Suppose the graph of f is given. Write equations (in terms of f) for the graphs that are obtained from the graph of f as follows.

(a) Shift 4 units upward.

(b) Shift 4 units downward.

(c) Shift 4 units to the right.

(d) Shift 4 units to the left.

(e) Reflect about the x-axis.

(f) Reflect about the y-axis.

(g) Stretch vertically by a factor of 3.

(h) Shrink vertically by a factor of 3.

34. Explain how the following graphs are obtained from the graph of $y = f(x)$.

(a) $y = 5f(x)$ **(b)** $y = f(x - 5)$

(c) $y = -f(x)$ **(d)** $y = -5f(x)$

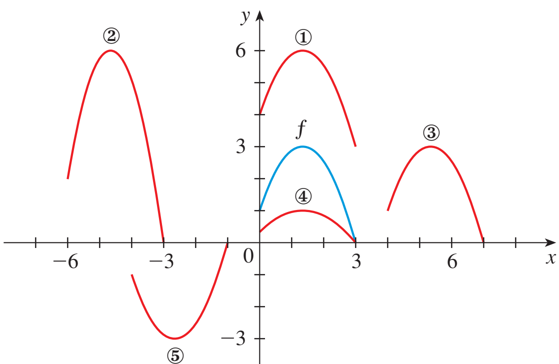
(e) $y = f(5x)$ **(f)** $y = 5f(x) - 3$

35. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

(a) $y = f(x - 4)$ **(b)** $y = f(x) + 3$

(c) $y = \frac{1}{3}f(x)$ **(d)** $y = -f(x + 4)$

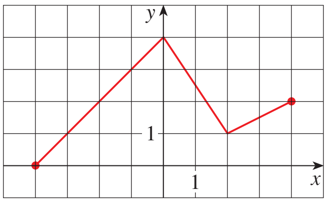
(e) $y = 2f(x + 6)$



36. The graph of f is given. Draw the graph of each of the following functions.

(a) $y = f(x + 4)$ **(b)** $y = f(x) + 4$

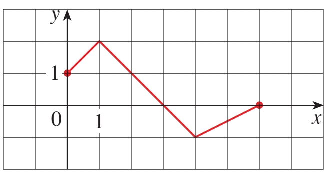
(c) $y = 2f(x)$ **(d)** $y = -\frac{1}{2}f(x) + 3$



37. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ **(b)** $y = f\left(\frac{1}{2}x\right)$

(c) $y = f(-x)$ **(d)** $y = -f(-x)$



38–42 ■ The graph of $y = \sqrt{x}$ is shown in Figure 8(a). Use transformations to graph each of the following functions.

38. $y = \sqrt{x} + 3$

39. $y = \sqrt{x + 3}$ **40.** $y = -\frac{1}{2}\sqrt{x}$

41. $y = -\sqrt{x - 1}$ **42.** $y = \sqrt{-x} + 2$

43–46 ■ The graph of $y = x^2$ is shown in Figure 9. Use transformations to graph each of the following functions.

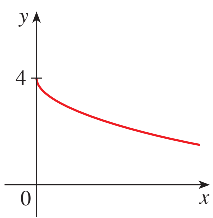
43. $y = -x^2 + 2$ **44.** $y = (x - 1)^2 - 4$

45. $f(x) = \frac{1}{4}x^2 - 3$

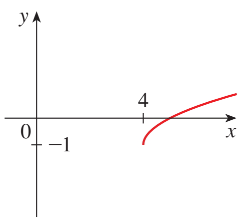
46. $g(x) = -(x + 5)^2 + 3$

47. Given the graph of $y = \sqrt{x}$ as shown in Figure 8(a), use transformations to create a function whose graph is as shown.

(a)

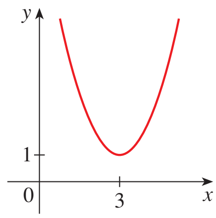


(b)

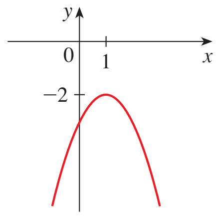


48. Given the graph of $y = x^2$ as shown in Figure 9, use transformations to create a function whose graph is as shown.

(a)



(b)



49. Water depth The depth, in feet, of water in a reservoir is given by $f(t)$, where t is the time in months beginning January 1, 2000.

(a) If a second reservoir's water depth is given by $g(t) = f(t) - 15$, how do the water levels of the two reservoirs compare?

(b) What if the second reservoir's depth is $g(t) = f(t - 2)$?

(c) What if the second reservoir's depth is $g(t) = f(t + 2)$?

(d) What if the second reservoir's depth is $g(t) = 0.8f(t)$?

50. Temperature The temperature, in degrees Fahrenheit, at Bob Hope Airport in California x days after the start of the year is given by $T(x)$.

(a) If the temperature x days after the start of the year at Los Angeles International Airport (LAX) is given by $h(x) = T(x) - 8$, how does the temperature at LAX compare to the temperature at Bob Hope Airport?

(b) What if the temperature at LAX is $h(x) = 0.9T(x)$?

51. Music sales The number of songs sold, in thousands, during the nth month of last year by an Internet music service is $A(n)$.

(a) If a rival service sold $B(n) = 1.3A(n)$ songs, how does the number of songs sold by the rival service compare to that of the first service?

(b) What if the rival service sold $B(n) = A(n) + 23$ songs?

(c) What if the rival service sold $B(n) = A(n - 1) + 5$ songs?

52. Bear population An ecologist has been observing the populations of brown bears and black bears in a region of Alaska. Let $R(t)$ represent the estimated number of brown bears, and $L(t)$ the estimated number of black bears, t years after January 1, 1990.

(a) If there are always 500 more black bears than brown bears, write a formula [in terms of $R(t)$] for $L(t)$.

(b) If there are always 15% fewer black bears than brown bears, write a formula [in terms of $R(t)$] for $L(t)$.

(c) If the number of black bears at any point in time matches the number of brown bears two years prior, write a formula [in terms of $R(t)$] for $L(t)$.

Motion A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.

(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f(g(t))$. What does this function represent?

Motion An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.

(b) Express the distance s between the plane and the radar station as a function of d.

(c) Use composition to express s as a function of t.

Water ripple A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Express the radius r of this circle as a function of the time t in seconds.

(b) If A is the area of this circle as a function of the radius, find $A(r(t))$ and interpret it.

56. **Electric current** The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

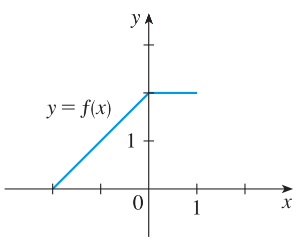
Challenge Yourself

58 ■ Find a formula for $p(x) = f(g(h(x)))$.

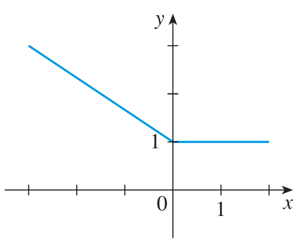
$f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 2, \quad h(x) = x + 3$

$f(x) = 2x - 1, \quad g(x) = x^2, \quad h(x) = 1 - x$

The graph of a function $y = f(x)$ is given.

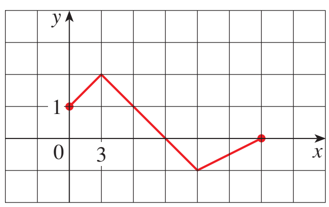


Write an equation (in terms of f) for the function whose graph is as shown.

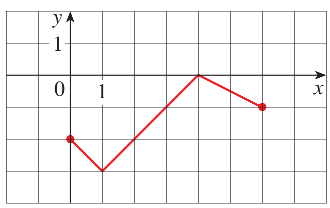


60–61 ■ If f is the graph given in Exercise 37, write a formula (in terms of f) for the function whose graph is shown.

60.

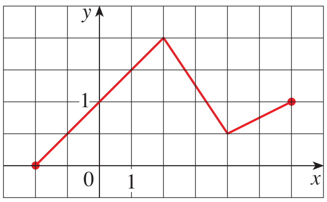


61.

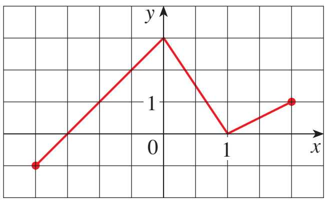


62–63 ■ If f is the graph given in Exercise 36, write a formula (in terms of f) for the function whose graph is shown.

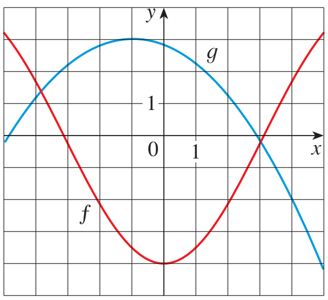
62.



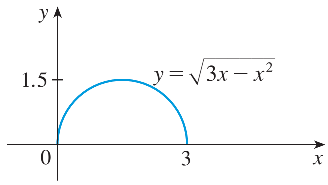
63.



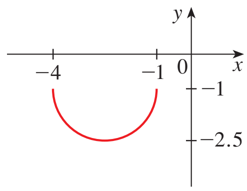
64. Use the given graphs of f and g to estimate the value of $h(x) = f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of h.



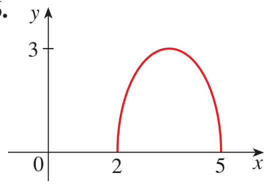
65–66 ■ The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



65.



66.



67. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. If $h(x) = f(g(x))$, is h also a linear function? If so, what is the slope of its graph?

68. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find formulas for $A(A(x))$, $A(A(A(x)))$, and $A(A(A(A(x))))$. What do these compositions represent? Find a formula for the composition of n copies of A.

1.3 Linear Models and Rates of Change

Of the many different types of functions that can be used to model relationships observed in the real world, one of the most common is the *linear function*. When we say that one quantity is a linear function of another, we mean that the graph of the function is a line.

Review of Lines

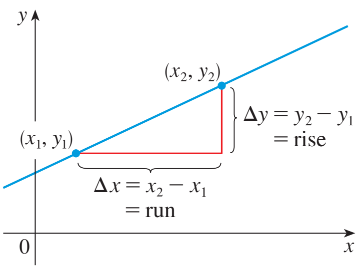


FIGURE 1

Recall that the *slope* of a line is a measure of its steepness. We measure the slope by computing the "rise over run" between any two points on the line:

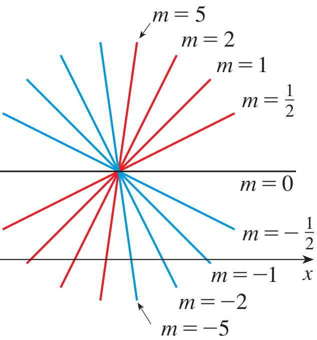
$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

As we can see in Figure 1, the rise is simply the difference or change in y-values between the two points and the run is the difference in x-values. Thus we can think of the slope as the "change in y over the change in x."

Greek letter Δ (capital delta) is
l to represent an increment or
mount of change.

(1) ■ Definition The **slope** of the line that passes through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



URE 2

A line has the same slope everywhere, so it makes no difference which two points we use to compute slope. Figure 2 shows several lines labeled with their slopes. Lines with positive slope slant upward to the right, whereas lines with negative slope slant downward to the right. Notice that the steepest lines are the ones for which the absolute value of the slope is largest, and a horizontal line has slope 0. The slope of a vertical line is not defined.

Now let's find an equation of the line that passes through a given point (x_1, y_1) and has slope m. If we compute the slope from (x_1, y_1) to any other point (x, y) on the line, we get

$$m = \frac{y - y_1}{x - x_1}$$

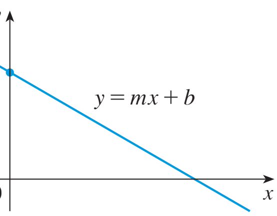
which can be written in the form

$$y - y_1 = m(x - x_1)$$

This equation is satisfied by all points on the line, including (x_1, y_1), and *only* by points on the line. Therefore it is an equation of the given line.

(2) ■ Point-Slope Form of the Equation of a Line An equation of the line passing through the point (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1)$$



URE 3

Equation 2 becomes even simpler if we use the point at which a (nonvertical) line intersects the y-axis. The x-coordinate there is 0 and the y-value, called the ***y*-intercept**, is traditionally denoted by b. (See Figure 3.) Thus the line passes through the point $(0, b)$ and Equation 2 becomes

$$y - b = m(x - 0)$$

which simplifies to the following:

(3) ■ Slope-Intercept Form of the Equation of a Line An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

■ EXAMPLE 1 A Line through Two Points

Find an equation of the line through the points $(-1, 2)$ and $(3, -4)$ and write the equation in slope-intercept form.

SOLUTION

By Definition 1 the slope of the line is

$$m = \frac{-4 - 2}{3 - (-1)} = -\frac{3}{2}$$

Using Equation 2 with $x_1 = -1$ and $y_1 = 2$, we obtain

$$y - 2 = -\tfrac{3}{2}(x + 1)$$

which can be written as

$$y - 2 = -\tfrac{3}{2}x - \tfrac{3}{2} \qquad \text{or} \qquad y = -\tfrac{3}{2}x + \tfrac{1}{2}$$

See Appendix B for a more detailed review of slope and lines, along with additional examples and exercises.

The equation of a nonvertical line can always be written in slope-intercept form, which reveals the slope and y-intercept at a glance.

■ EXAMPLE 2 Graphing a Linear Equation

Sketch the graph of the equation $y = -\frac{3}{4}x + 5$.

SOLUTION

The equation is in slope-intercept form, which allows us to identify the graph as a line with slope $m = -\frac{3}{4}$ and y-intercept 5. To sketch the line we start at the point (0, 5). The slope is negative, so we move through a "rise" of -3 (actually a downward movement) and a run of 4 (to the right) to arrive at the point (4, 2). The graph is the line through these two points as shown in Figure 4.

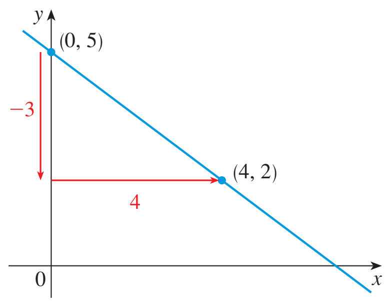


FIGURE 4

■ Rate of Change and Linear Functions

We defined the slope of a line as the ratio of the change in y, Δy, to the change in x, Δx. Thus we can interpret the slope as the **rate of change** of y with respect to x. If f is a linear function, then its graph is a line and we can think of the slope as the ratio of the change in output to the change in input. In this context, the slope measures the rate of change of the function.

$$\text{rate of change} = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \text{slope of line}$$

For instance, a slope of 4 means that a change in input will cause a change in output that is four times larger. The slope of a given line is the same at all points, so a characteristic feature of linear functions is that *the rate of change is constant*:

Linear functions grow at a constant rate.

A rate of change is always measured by a ratio of units: output units per input unit.

EXAMPLE 3 Slope of a Linear Function

A company that produces snowboards has seen its annual sales increase linearly. In 2005, it sold 31,300 snowboards, and it sold 38,200 snowboards in 2011. Compute the slope of the linear function that gives annual sales as a function of the year. What does the slope represent in this context?

SOLUTION

The slope is

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{38{,}200 - 31{,}300}{2011 - 2005} = \frac{6900}{6} = 1150$$

and the units are number of snowboards per year. Thus the number of snowboards the company produces is increasing at a rate of 1150 per year. ■

Because the graph of a linear function is a line, we can write an equation for a linear function using the slope-intercept form given by Equation 3:

$$f(x) = mx + b$$

x	$f(x)$
-10	22
-5	25
0	28
5	31
10	34
15	37

For instance, consider a linear function with values as given in the table. Observe that each time x increases by 5, $y = f(x)$ increases by 3, so the slope is $\frac{3}{5}$. We have $f(0) = 28$, so the point $(0, 28)$ is on the line and 28 is the y-intercept. (If the dependent variable is A rather than y, we would call it the A-intercept.) Thus an equation for f is $f(x) = \frac{3}{5}x + 28$.

If f is measuring some quantity, we can think of 28 (when $x = 0$) as the initial or starting value for the function. Values change from there at a rate of $\frac{3}{5}$. The next two examples illustrate this point.

EXAMPLE 4 A Linear Cost Function

The owner of a car-wash business estimates that it costs $C(x) = 4.5x + 340$ dollars to wash x cars in one day.

(a) What is the rate of change? What does it mean in this context?

(b) What is the C-intercept? What does it represent here?

SOLUTION

(a) The rate of change is 4.5, the coefficient of x, and the units are dollars per car. Thus each additional car washed adds \$4.50 to the cost for that day.

(b) The C-intercept is 340. This value is the initial output corresponding to $x = 0$, so it represents the fixed cost, \$340, of operating the car wash for a day, whether or not any cars are washed. ■

■ EXAMPLE 5 Writing an Equation for a Linear Function

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent in this context?

(c) What is the temperature at a height of 2.5 km?

SOLUTION

(a) Because we are assuming that T is a linear function of h, we can write

$$T = mh + b$$

We are given two function values, $T(0) = 20$ and $T(1) = 10$, so the slope of the graph is

$$m = \frac{\Delta T}{\Delta h} = \frac{T(1) - T(0)}{1 - 0} = \frac{10 - 20}{1 - 0} = -10$$

The initial temperature value is $T(0) = 20$, so the T-intercept is $b = 20$ and the linear function is

$$T(h) = -10h + 20$$

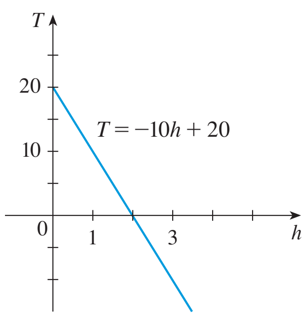


FIGURE 5

(b) The graph is sketched in Figure 5. The slope is $m = -10$ and it represents the rate of change of T with respect to h; the units are degrees Celsius per kilometer (°C/km). The rate of change is negative, so the temperature decreases by 10°C for each rise in elevation of 1 km.

(c) At a height of $h = 2.5$ km, the temperature is

$$T(2.5) = -10(2.5) + 20 = -5°\text{C}$$ ■

We don't need to know the starting value for a linear function to write an equation. We can simply use the point-slope formula given in Equation 2.

■ EXAMPLE 6 Writing a Linear Model Using the Point-Slope Form

A pump has been pouring water into a swimming pool. The data in the table show the water volume of the pool every two hours after the pump was activated.

Hours	Water volume in pool (gallons)
2	2800
4	3100
6	3400
8	3700
10	4000

(a) Explain why a linear model is appropriate.

(b) Write an equation for a linear function to model the data.

(c) Use your model to predict the volume of water in the pool after 17.5 hours.

(d) When will the amount of water in the pool reach 6000 gallons?

SOLUTION

(a) The volume of water increases 300 gallons during each two-hour interval. Thus the rate of change is constant, indicating a linear relationship between the input and output.

(b) The rate of change is 300 gallons every two hours, or 150 gallons per hour, so the slope is $m = 150$. If we let $V(t)$ be the water volume in gallons t hours after the pump is activated, then $V(2) = 2800$, so the point (2, 2800) is on the graph. The point-slope formula gives

$$V - 2800 = 150(t - 2) = 150t - 300$$
$$V = 150t + 2500$$

Thus the water volume in gallons t hours after the pump is turned on is given by $V(t) = 150t + 2500$.

(c) The volume of water in the pool after 17.5 hours is

$$V(17.5) = 150(17.5) + 2500 = 5125 \text{ gallons}$$

(d) We solve $V(t) = 6000$:

$$150t + 2500 = 6000$$
$$150t = 3500$$
$$t = \frac{3500}{150} = \frac{70}{3}$$

Thus the amount of water in the pool reaches 6000 gallons after $23\frac{1}{3}$ hours, or 23 hours 20 minutes. ■

Fitting a Model to Data

Although linear functions are often used as models, few functional relationships in the world around us are perfectly linear. Nevertheless, many observed data are *approximately* linear, and a linear function is still an appropriate model to use. The goal is to write an equation that "fits" the data accurately enough to capture the basic trend and allow further analysis. But how do we know if an equation fits the data well enough? The next example illustrates one approach we can take.

■ EXAMPLE 7 Modeling with a Linear Function

Table 1 on page 35 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2008. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION

We use the data in Table 1 to make the scatter plot in Figure 6, where t represents time, in years, and C represents the CO_2 level, in parts per million (ppm). To simplify the input values, let $t = 0$ correspond to the year 1980.

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One strategy is to use two of the points from the scatter plot to write an equation. Different pairs of points will generate different results, so the points should be chosen wisely. You may wish to first draw a line with a ruler on the scatter plot to help select two points.

TABLE 1

Year	CO_2 level (in ppm)
1980	338.7
1982	341.2
1984	344.4
1986	347.2
1988	351.5
1990	354.2
1992	356.3
1994	358.6
1996	362.4
1998	366.5
2000	369.4
2002	373.2
2004	377.5
2006	381.9
2008	385.6

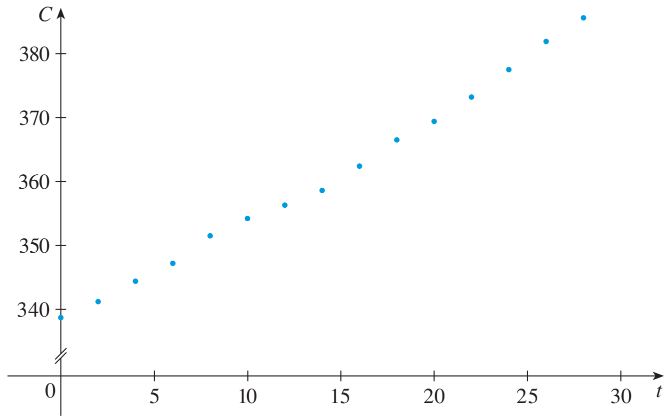


FIGURE 6 Scatter plot for the average CO_2 level

From the scatter plot, it appears that a line passing through the points (4, 344.4) and (20, 369.4) will fit the data reasonably well. The slope of this line is

$$\frac{369.4 - 344.4}{20 - 4} = \frac{25.0}{16} = 1.5625$$

and its equation is

$$C - 344.4 = 1.5625(t - 4)$$

or

(4) $$C = 1.5625t + 338.15$$

Equation 4 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 7.

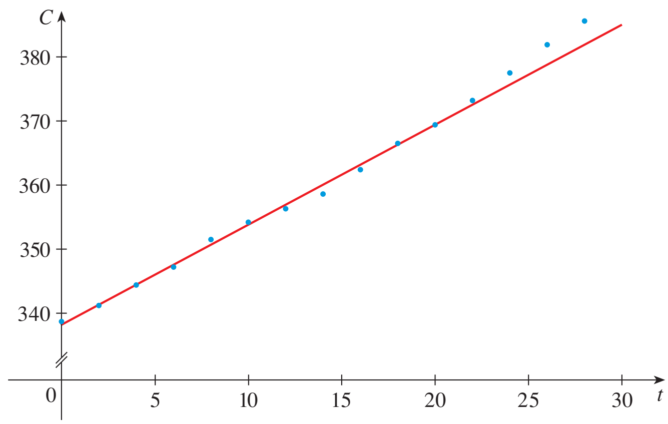


FIGURE 7
Linear model through two data points

Regression Lines

A more sophisticated procedure for finding a linear model to fit data is called the *method of least squares*. In this method, the vertical distance from each data point to a line is measured, and the squares of the distances are added together. Of all possible lines, the line that gives the smallest of such sums, called the **regression line**, is chosen as the best-fitting line. This process is tedious to carry out by hand, but careful not to round the values model equation too much. don't need to include all the mal places that a calculator omputer gives, but rounding es to too few digits can greatly ease the accuracy of the model.

computer software and most graphing calculators can determine the equation with ease. Using a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. The calculator gives the slope and y-intercept of the regression line as approximately

$$m = 1.6543 \qquad b = 337.41$$

So the regression line model for the CO_2 level is

$$C = 1.6543t + 337.41 \tag{5}$$

In Figure 8 we graph the regression line as well as the data points. Comparing with Figure 7, we see that it gives a better fit than our first linear model.

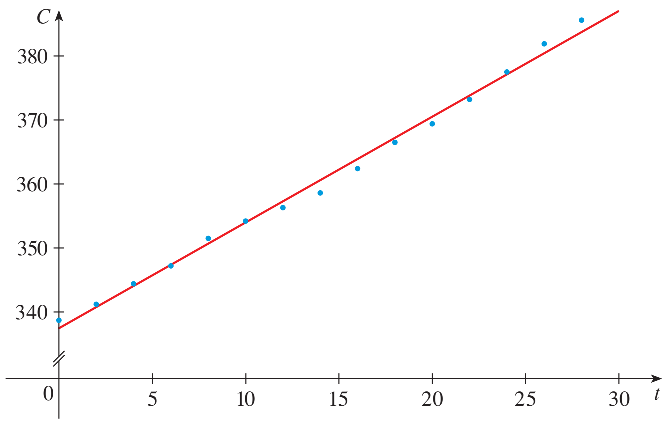


FIGURE 8
The regression line

Notice that here the regression line does not include *any* of the original data points. In many cases, the best-fitting line is not one that passes through data points.

Interpolation and Extrapolation

Now that we have found a linear model for the carbon dioxide levels, we can use it to estimate the CO_2 levels for years not listed in Table 1. If we use the model to compute values for years between the years in the table, we are **interpolating**. In contrast, **extrapolation** is the computation of values outside given data. In general, extrapolation carries a higher risk of inaccuracy because it is often hard to predict whether an observed trend will continue.

■ EXAMPLE 8 Using a Linear Model to Interpolate and Extrapolate

Use the linear model given by Equation 5 to estimate the average CO_2 level for 1987 and to predict the level for the year 2016. According to this model, when will the average annual CO_2 level exceed 410 parts per million?

SOLUTION

The year 1987 corresponds to $t = 7$. Using Equation 5, we estimate that the average CO_2 level in 1987 was

$$C(7) = (1.6543)(7) + 337.41 \approx 348.99 \text{ ppm}$$

This is an example of interpolation because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average

To predict the level for the year 2016, we use :

$$C(36) = (1.6543)(36) + 337.41 \approx 396.96 \text{ ppm}$$

So we predict that the average CO_2 level in the year 2016 will be 396.96 ppm. This is an example of extrapolation because we have predicted a value *outside* the observed years 1980–2008. Consequently, we are far less certain about the accuracy of our prediction.

To determine when our model predicts a CO_2 level of 410 ppm, we solve $C(t) = 410$:

$$1.6543t + 337.41 = 410$$

$$1.6543t = 72.59$$

$$t = \frac{72.59}{1.6543} \approx 43.88$$

The solution corresponds to a value between the years 2023 and 2024, and since the CO_2 levels are annual averages, only integer inputs give meaningful function values. Comparing $C(43) \approx 408.54$ and $C(44) \approx 410.20$, we see that the CO_2 level will first exceed an annual average of 410 ppm in 2024. Note that this prediction is somewhat risky because it involves a time quite a few years beyond the observed values, and no one can say whether the same patterns will continue. ■

In the preceding example, we used a continuous function to model discrete data. We will often find this to be a useful technique. However, we must use caution in interpreting results found using the continuous model, as the example illustrates.

Exercises 1.3

1–4 ■ Find the slope of the line through the given points.

1. (3, 7), (5, 10)

2. (1, 8), (4, −6)

3. (45, 1860), (26, 2240)

4. (185, 5600), (210, 8150)

5–14 ■ Find an equation of the line that satisfies the given conditions. Express the equation in slope-intercept form.

5. Slope 3, y-intercept −2

6. Slope $\frac{2}{5}$, y-intercept 4

7. Through (2, −3), slope 6

8. Through (−3, −5), slope $-\frac{7}{2}$

9. Through (2, 1) and (1, 6)

10. Through (−1, −2) and (4, 3)

11. Through (4, 84) and (13, −312)

12. Through (6, 70) and (16, 300)

14. x-intercept −8, y-intercept 6

15. Sketch a line through the point (−2, 6) with slope $-\frac{1}{5}$.

16. Sketch a line with slope $\frac{4}{7}$ and y-intercept −3.

17–20 ■ Find the slope and y-intercept of the line. Then draw its graph.

17. $2x + 5y = 15$

18. $3x - 8y = -10$

19. $-5x + 6y = 42$

20. $8y - 3x = 48$

21–24 ■ Find the slope and intercepts of the linear function. Then sketch a graph.

21. $f(x) = -2x + 14$

22. $g(t) = -0.5t + 5$

23. $A(t) = 0.2t - 4$

24. $P(v) = 3v - 1$

25. Write an equation for a linear function h where $h(7) = 329$ and $h(11) = 553$.

26. Write an equation for a linear function w where

Television ratings The weekly ratings, in millions of viewers, of a recent television program are given by $L(w)$, where w is the number of weeks since the show premiered. If L is a linear function where $L(8) = 5.32$ and $L(12) = 8.36$, compute the slope of L and explain what it represents in this context.

Consumer demand A movie studio is releasing a new DVD, and the studio estimates that if the DVD is priced at \$19.99, it will sell 6.68 million copies, whereas if it is priced at \$15.99, it will sell 11.27 million copies. If f is a linear function that gives the number of copies sold (in millions) at a given price, what is the slope of f? What does it represent in this context?

Depreciation A small company purchased a new copy machine for \$16,500 and the company's accountant plans to depreciate (for tax purposes) the machine to a value of \$0 over five years. If $V(t)$ is the value of the machine after t years, and V is a linear function, what is the slope of V? What does the slope represent in this context?

Matric suction Matric suction is the pressure that causes water to flow from wetter soil to dryer soil and often decreases linearly with depth. A researcher has taken samples at a location at various depths. Let $g(d)$ be the matric suction, measured in kilopascals (kPa), at a depth of d cm. Assuming that g is a linear function, if $g(35) = 78$ and $g(104) = 43$, what is the slope of g? What does the slope represent in this context?

Taxes Suppose the taxes a company pays are approximately $T(p) = 0.26p + 15.4$ thousand dollars, where p is the company's annual profit in thousands of dollars. What is the rate of change, and what does it measure in this context?

Earth's surface temperature Some scientists believe that the average surface temperature of the earth has been rising steadily. They have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What are the slope and T-intercept? What do they represent in this context?

(b) Use the equation to predict the average global surface temperature in 2100.

Drug dosage If the recommended adult dosage for a drug is D, in mg, then to determine the appropriate dosage c for a child of age a, pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.

(a) Find the slope of the graph of c (as a function of a). What does it represent?

(b) What is the dosage for a newborn?

Consumer demand The manager of a weekend flea market knows from past experience that if he charges

number y of spaces he can rent is given by the equation $y = 200 - 4x$.

(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)

(b) What do the slope, the y-intercept, and the x-intercept of the graph represent?

35. Temperature scales The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

(a) Sketch a graph of this function.

(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

36. Driving distance Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-96. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.

(a) Express the distance traveled in terms of the time elapsed.

(b) Draw the graph of the equation in part (a).

(c) What is the slope of this line? What does it represent?

37. Cricket chirping rate Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.

(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.

(b) What is the slope of the graph? What does it represent?

(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

38. Manufacturing cost The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.

(a) Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.

(b) What is the slope of the graph and what does it represent?

(c) What is the y-intercept of the graph and what does it represent?

39. Ocean water pressure At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in^2. Below the surface, the water pressure increases by 4.34 lb/in^2 for every 10 ft of descent.

(a) Express the water pressure as a function of the depth below the ocean surface.

(b) At what depth is the pressure 100 lb/in^2?

40. Car expense The monthly cost of driving a car depends cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.

(a) Express the monthly cost C as a function of the distance driven d, assuming that a linear relationship gives a suitable model.

(b) Use part (a) to predict the cost of driving 1500 miles per month.

(c) Draw the graph of the linear function. What does the slope represent?

(d) What does the C-intercept represent?

(e) Why does a linear function give a suitable model in this situation?

41. Ulcer rates The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph a linear model using the third and last data points.

(c) According to the model, how likely is someone with an income of \$90,000 to suffer from peptic ulcers? Is this an example of interpolation, or extrapolation?

(d) Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

42. US public debt The table gives the amount of debt held by the public in the United States (at the end of the year) as estimated by the Congressional Budget Office of the US federal government.

Year	Public debt (billions of dollars)
2004	4296
2005	4665
2006	4971
2007	5246
2008	5494
2009	5716
2010	5919

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph a linear model using the second and sixth data points.

(c) According to the model, when will the public debt reach \$5.1 trillion? Is this an example of interpolation, or extrapolation?

(d) Use the model to predict the projected public debt in 2014.

43. Vehicle value In general, a used car is worth more if it has low mileage. The table shows how the value of a particular vehicle is affected by different mileage figures.

Mileage	Value
20,000	\$14,245
30,000	\$13,520
40,000	\$12,520
50,000	\$11,645
60,000	\$10,970

(a) Make a scatter plot of these data and use two of the data points to write a linear model for the data.

(b) Use the model to estimate the value of the same car if it has been driven only 12,000 miles.

(c) For how many miles driven does the model predict a value of \$0? Is this realistic?

44. Hospital visits The Center for Disease Control compiles the average annual hospital emergency room visits due to falls for children of various ages. Annual averages during recent years are listed in the table.

Age	Number of visits per 10,000 population
9	295.4
11	276.1
13	268.1
15	215.8
17	186.0
19	176.5

(a) Make a scatter plot of these data. Is a linear model appropriate?

(b) Use two of the data points to write a linear model for the data.

(c) Use the model to estimate the average number of emergency room visits per 10,000 population for 18-year-olds. How does your estimated value compare

45. **Ulcer rates** Exercise 41 lists data for peptic ulcer rates for various family incomes.

(a) Find the least squares regression line for these data.

(b) Use the linear model in part (a) to estimate the ulcer rate for an income of \$25,000.

(c) According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?

46. **Cricket chirping rates** Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

(a) Make a scatter plot of the data.

(b) Find and graph the regression line.

(c) Use the linear model in part (b) to estimate the chirping rate at 100°F.

47. **Olympics** The table gives the winning heights for the men's Olympic pole vault competitions up to the year 2004.

Year	Height (m)	Year	Height (m)
1896	3.30	1960	4.70
1900	3.30	1964	5.10
1904	3.50	1968	5.40
1908	3.71	1972	5.64
1912	3.95	1976	5.64
1920	4.09	1980	5.78
1924	3.95	1984	5.75
1928	4.20	1988	5.90
1932	4.31	1992	5.87
1936	4.35	1996	5.92
1948	4.30	2000	5.90
1952	4.55	2004	5.95
1956	4.56		

(a) Make a scatter plot and decide whether a linear model is appropriate.

(b) Find and graph the regression line.

(c) Use the linear model to predict the height of the winning pole vault at the 2008 Olympics and compare with the actual winning height of 5.96 meters.

(d) Is it reasonable to use the model to predict the winning height at the 2100 Olympics?

48. **US public debt** The table gives the projected amount of debt held by the public in the United States as estimated by the Congressional Budget Office.

Year	Public debt (billions of dollars)
2011	6012
2012	5955
2013	5884
2014	5784
2015	5658

(a) Make a scatter plot of these data and decide whether a linear model is appropriate.

(b) Find and graph the least squares regression line.

(c) Use the regression line model to predict the projected public debt in 2020.

(d) Exercise 42 lists similar data for prior years, and in part (d) the public debt is estimated for 2014. How does your predicted value compare to the value given here? What can you conclude about extrapolation?

49. A *family of functions* is a collection of functions whose equations are related.

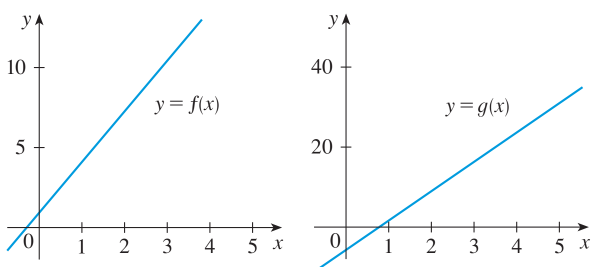
(a) What do all members of the family of linear functions $f(x) = 3x + c$ have in common? Sketch several members of the family.

(b) What do all members of the family of linear functions $f(x) = ax + 3$ have in common? Sketch several members of the family.

(c) Which function belongs to both families?

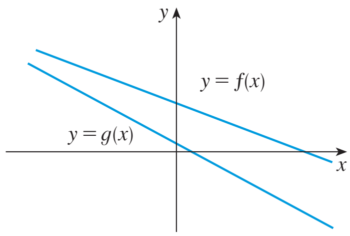
50. What do all members of the family of linear functions $f(x) = c - (x + 3)$ have in common? Sketch several members of the family.

51. Two linear functions are graphed below. Which function has the greater rate of change?



Graphing calculator or computer required

52. Two linear functions $f(x) = ax + b$ and $g(x) = cx + d$ are graphed below.



(a) Is $a > c$, or is $a < c$?

(b) Is $b > d$, or is $b < d$?

53. Health spending The table shows the US national health expenditures, as a percentage of gross domestic product (GDP), for various years.

Year	National health expenditure (percent of GDP)
1995	13.4
1998	13.2
1999	13.2
2000	13.3
2001	14.1
2002	14.9
2003	15.3

Write a piecewise function that models these data.

54. US public debt Use the data given in Exercises 42 and 48 to write a piecewise function that models the estimated public debt for the years 2004–2015.

1.4 Polynomial Models and Power Functions

While linear functions may be the most basic and commonly occurring models, there are many other kinds of functions that are often used in modeling. In this section we look at polynomials, power functions, and rational functions. We will see that there is some overlap between the categories; some functions qualify as all three types.

A function P is called a **polynomial** if it can be written as

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. Examples of polynomials are

$$f(x) = x^5 - 3x^8 + 4x^2$$

$$g(t) = 1.737t^3 - 2.49t^2 + 8.51t + 4.12$$

$$P(v) = 2v^6 - v^4 + \tfrac{2}{5}v^3 + \sqrt{2}$$

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. The largest exponent that appears on the input variable is called the **degree** of the polynomial. The degrees of the polynomials above are 8, 3, and 6, respectively. The linear functions $f(x) = mx + b$ studied in the previous section are polynomials of degree 1. (Recall that $x^1 = x$.)

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, we will soon see why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity.

5 Exponential Models

You may have heard a newscaster say that a company or industry was "growing exponentially." Populations and financial markets can also grow exponentially. What does this mean? Suppose a bacteria population (cultured in ideal conditions) is doubling every hour. If we begin with 1000 bacteria, then after one hour we have $1000 \times 2 = 2000$ bacteria, after two hours we have $2000 \times 2 = 4000$ bacteria, after three hours we have 8000 bacteria, and so on. The population is not growing in a linear fashion: If we had a constant rate of change, the population would consistently increase by 1000 every hour. Instead, we have a constant *percentage* rate of growth; the population increases by 100% every hour.

To model the growth of the bacteria population, let $p(t)$ be the number of bacteria after t hours. Then

$$\begin{aligned} p(0) &= 1000 \\ p(1) &= 2 \times p(0) = 2 \times 1000 \\ p(2) &= 2 \times p(1) = 2 \times (2 \times 1000) = 2^2 \times 1000 \\ p(3) &= 2 \times p(2) = 2 \times (2^2 \times 1000) = 2^3 \times 1000 \end{aligned}$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = 1000(2^t)$$

Our result is a constant multiple of the *exponential function* $y = 2^t$. It is called an exponential function because the variable, t, is the exponent. It should not be confused with the power function $y = t^2$.

Introduction to Exponential Functions

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant called the **base**. Every exponential function has domain $\mathbb{R} = (-\infty, \infty)$, although this may not be immediately apparent. For instance, if we input a positive integer such as 8, then we simply have

$$f(8) = a^8 = \underbrace{a \cdot a \cdot \cdots \cdot a}_{\text{8 factors}}$$

If we input a negative integer such as -3, then recall that

$$f(-3) = a^{-3} = \frac{1}{a^3}$$

We can input 0 as well: $f(0) = a^0 = 1$. Reciprocal inputs such as $1/3$ become roots:

$$f\left(\tfrac{1}{3}\right) = a^{1/3} = \sqrt[3]{a}$$

In fact, any rational number input x can be expressed as a fraction p/q (where p and

q are integers), in which case

$$f(x) = f(p/q) = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

What if x is an irrational number, like $\sqrt{2}$? We can define $a^{\sqrt{2}}$ by a limiting process using rational approximations to $\sqrt{2}$. Because $\sqrt{2}$ can be approximated by 1.4, 1.41, 1.414, 1.4142, . . . with increasing accuracy, so $a^{\sqrt{2}}$ is approximated by $a^{1.4}$, $a^{1.41}$, $a^{1.414}$, $a^{1.4142}$, ... For our purposes, it is enough to know that a calculator can generate an (approximate) value. Thus a^x is defined for any real number input.

The range of all exponential functions (except $1^x = 1$) is $(0, \infty)$. (An exponential function can never output 0 or a negative number.)

The graph of $y = 2^x$ is shown in Figure 1 and the graphs of members of the family of functions $y = a^x$ are shown in Figure 2 for various values of the base a. Notice that all of these graphs pass through the same intercept point (0, 1) because $a^0 = 1$ for all positive values of a. Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

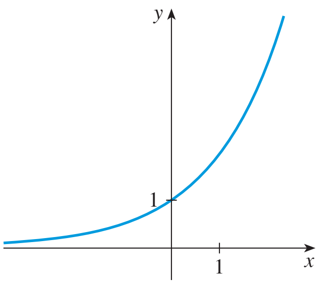


FIGURE 1
$y = 2^x$

If $0 < a < 1$, then a^x approaches 0 as x becomes large. If $a > 1$, then a^x approaches 0 as x decreases through negative values. In both cases the x-axis is a horizontal asymptote. These matters are discussed in Section 4.4.

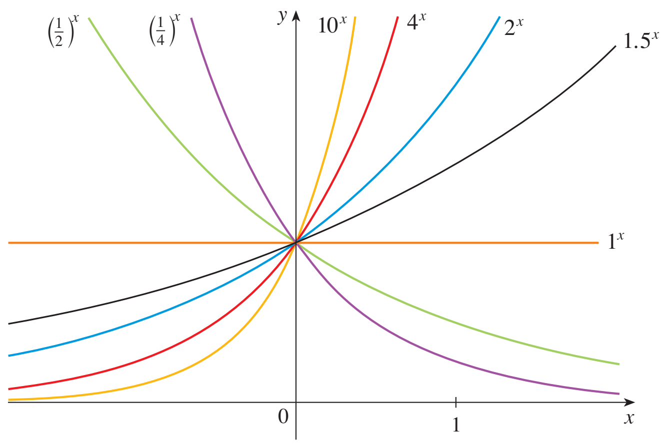


FIGURE 2

You can see from Figure 2 that there are basically two kinds of exponential functions $y = a^x$ (assuming $a \neq 1$). If $0 < a < 1$, the exponential function decreases; if $a > 1$, it increases. These cases are illustrated in Figure 3. Notice that, since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y-axis.

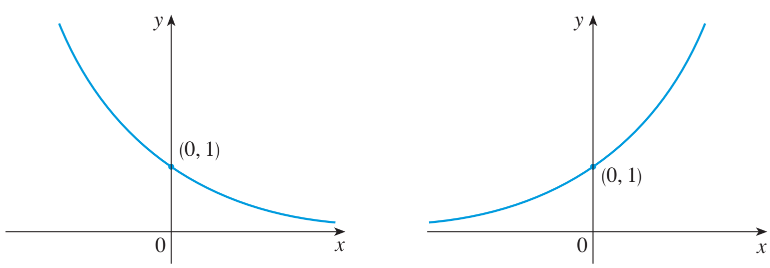


FIGURE 3 (a) $y = a^x$, $0 < a < 1$ (b) $y = a^x$, $a > 1$

■ EXAMPLE 1 Sketching Graphs of Exponential Functions

Sketch the graphs of the functions **(a)** $f(x) = 3 \cdot 2^x$ and **(b)** $g(x) = \left(\frac{1}{2}\right)^x + 3$. What are the domain and range?

SOLUTION

a review of reflecting and shift- graphs, see Section 1.2.

(a) The graph of f (shown in Figure 4) is the graph of $y = 2^x$ (see Figure 1) stretched vertically by a factor of 3. The graph intersects the y-axis at (0, 3) but the domain, range, and horizontal asymptote remain unchanged.

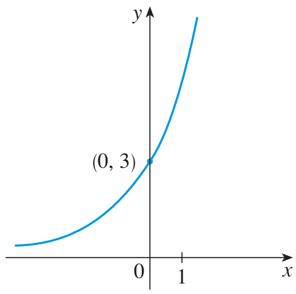


FIGURE 4

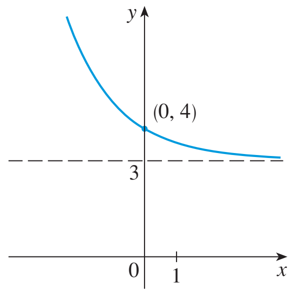


FIGURE 5

(b) The graph of $y = (1/2)^x$ is shown in Figure 2. We shift the graph upward 3 units to obtain the graph of g. (See Figure 5.) The y-intercept is shifted to 4 and the horizontal asymptote is the line $y = 3$. The domain is $\mathbb{R}$ and the range is $(3, \infty)$. ■

■ EXAMPLE 2 Comparing Exponential and Power Functions

Use a graphing calculator (or computer) to compare the exponential function $f(x) = 2^x$ and the power function $g(x) = x^2$. Which function grows more quickly when x is large?

SOLUTION

Figure 6 shows both functions graphed in the viewing rectangle $[-2, 6]$ by $[0, 40]$. We see that the graphs intersect three times, but for $x > 4$ the graph of $f(x) = 2^x$ stays above the graph of $g(x) = x^2$. Figure 7 gives a more global view and shows that, for large values of x, the exponential function $y = 2^x$ grows far more rapidly than the power function $y = x^2$.

mple 2 shows that $y = 2^x$ eases more quickly than $y = x^2$. demonstrate just how quickly $= 2^x$ increases, let's perform following thought experiment. pose we start with a piece of er a thousandth of an inch thick we fold it in half 50 times. h time we fold the paper in half, thickness of the paper doubles, he thickness of the resulting er would be $2^{50}/1000$ inches. v thick do you think that is? It ks out to be more than 17 mil- miles!

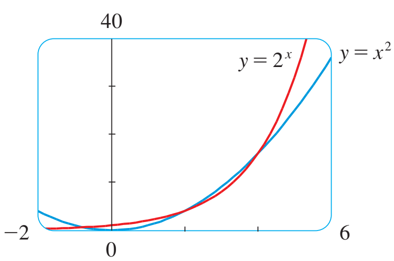


FIGURE 6

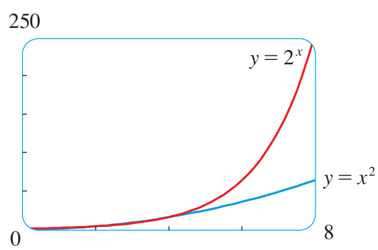


FIGURE 7 ■

Properties of Exponential Functions

One reason for the importance of the exponential function lies in the following properties. If x and y are integers or rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for all real numbers x and y.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

1. $a^x \cdot a^y = a^{x+y}$ **2.** $\dfrac{a^x}{a^y} = a^{x-y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

For more review and practice using the Laws of Exponents, see Appendix A.

EXAMPLE 3 Using Properties of Exponential Functions

Show that each of the following is true.

(a) $8 \cdot (1.6)^{2x} = 8 \cdot (2.56)^x$

(b) $5 \cdot 4^{x/2} = 5 \cdot 2^x$

(c) $\dfrac{10}{5^{x/3}} = 10 \cdot (5^{-1/3})^x$

(d) $3^{4+2t} = 81 \cdot 9^t$

SOLUTION

(a) $8 \cdot (1.6)^{2x} = 8 \cdot ((1.6)^2)^x = 8 \cdot (2.56)^x$

(b) $5 \cdot 4^{x/2} = 5 \cdot (4^{1/2})^x = 5 \cdot \left(\sqrt{4}\right)^x = 5 \cdot 2^x$

(c) $\dfrac{10}{5^{x/3}} = 10 \cdot (5^{-x/3}) = 10 \cdot (5^{-1/3})^x$

(d) $3^{4+2t} = 3^4 \cdot 3^{2t} = 81 \cdot (3^2)^t = 81 \cdot 9^t$ ■

Applications of Exponential Functions

The exponential function occurs very frequently in mathematical models of nature and society. Any situation where a quantity is growing or shrinking at a constant percentage rate exhibits *exponential growth* or *exponential decay* and can be modeled with a transformed exponential function. Here we give an example where such a model is appropriate to describe population growth. In Section 3.6 we will study many additional applications.

Many graphing calculators (and computer software) have exponential regression capabilities that can fit an exponential model to data. They typically use a least squares technique similar to the linear regression method we used in Section 1.3. The following example uses this technology to model the world's human population over the last century.

■ EXAMPLE 4 Modeling Population with Exponential Regression

BLE 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot. For simplicity, we have used $t = 0$ to represent 1900.

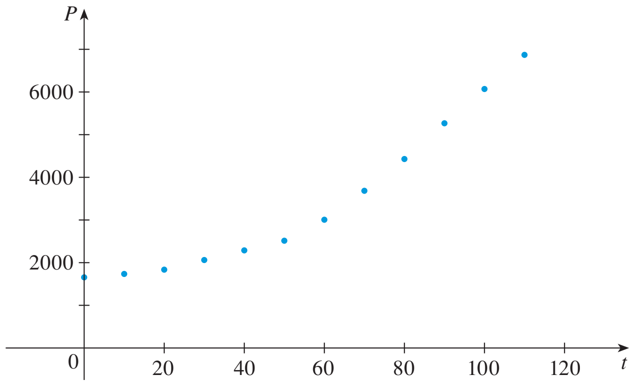


FIGURE 8 Scatter plot for world population growth

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator to obtain the exponential model

$$P(t) = (1436.53) \cdot (1.01395)^t$$

Figure 9 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

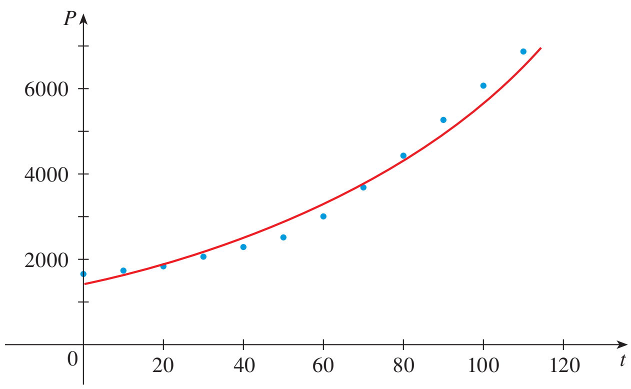


FIGURE 9 Exponential model for population growth

■

The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base a is influenced by the way the graph of $y = a^x$ crosses the y-axis. Figures 10 and 11 show the *tangent lines* to the graphs of $y = 2^x$ and $y = 3^x$ at the point $(0, 1)$. (Tangent lines will be defined precisely in Section 2.3. For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point. It has the same direction as the exponential graph at that point.) If we measure the slopes of these tangent lines at (0, 1), we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

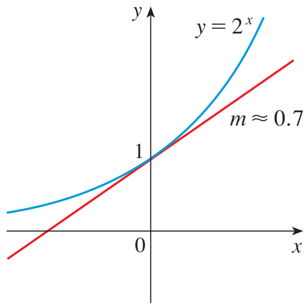


FIGURE 10

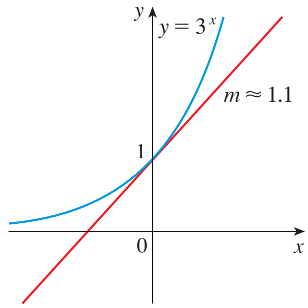


FIGURE 11

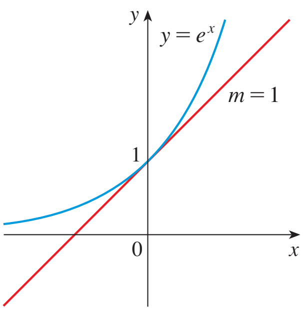


FIGURE 12
The natural exponential function crosses the y-axis with a slope of 1.

TEC Module 1.5 enables you to graph exponential functions with various bases and their tangent lines in order to estimate more closely the value of a for which the tangent has slope 1.

It turns out, as we will see in Chapter 3, that some of the formulas of calculus will be greatly simplified if we choose the base a so that the slope of the tangent line to $y = a^x$ at (0, 1) is *exactly* 1. (See Figure 12.) In fact, there *is* such a number; it is an irrational number (it has an infinite nonrepeating decimal representation) and is denoted by the letter e. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.) This value also arises naturally in the analysis of compounded interest on a bank account, as one example. In view of Figures 10 and 11, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 13.) We call $y = e^x$ the *natural exponential function*. In Chapter 3 we will see that the value of e, correct to five decimal places, is

$$e \approx 2.71828$$

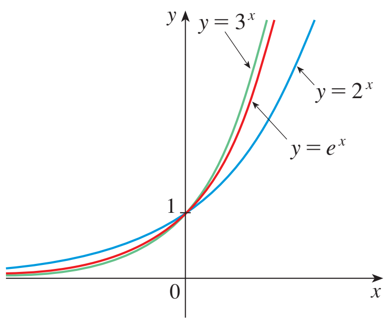


FIGURE 13

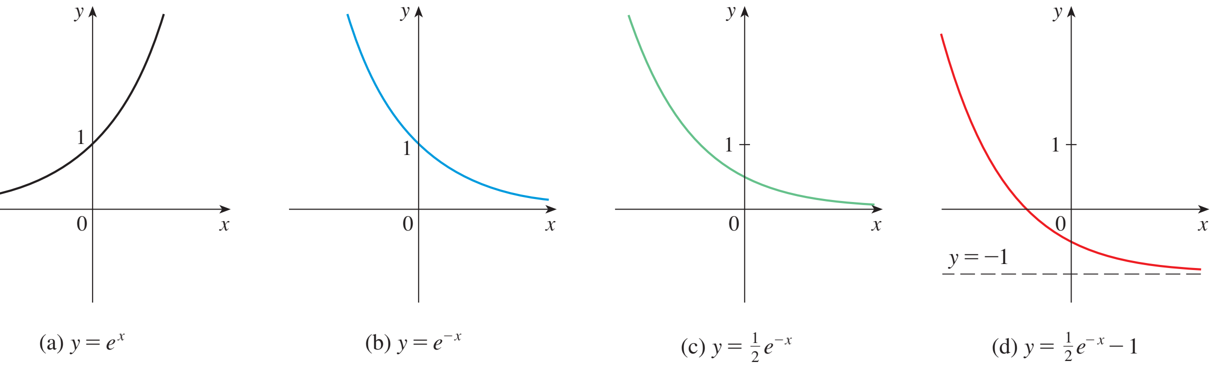
■ EXAMPLE 5

Graphing a Transformed Natural Exponential Function

Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

SOLUTION

We start with the graph of $y = e^x$ from Figures 12 and 14(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 14(b). (Notice that the graph crosses the y-axis with a slope of -1). Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 14(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 14(d). The y-intercept is $-\frac{1}{2}$ and the horizontal asymptote has shifted to $y = -1$. The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.



(a) $y = e^x$ (b) $y = e^{-x}$ (c) $y = \frac{1}{2}e^{-x}$ (d) $y = \frac{1}{2}e^{-x} - 1$

URE 14

How far to the right do you think we would have to go for the height of the graph of $y = e^x$ to exceed a million? The next example demonstrates the rapid growth of this function by providing an answer that might surprise you.

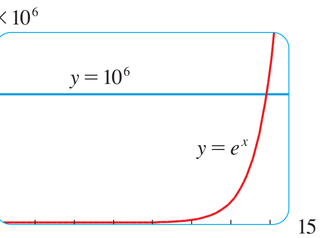
■ EXAMPLE 6

The Rapid Growth of the Natural Exponential Function

Use a graphing device to find the values of x for which $e^x > 1{,}000{,}000$.

SOLUTION

In Figure 15 we graph both the function $y = e^x$ and the horizontal line $y = 1{,}000{,}000$. We see that these curves intersect when $x \approx 13.8$. Thus, $e^x > 10^6$ when $x > 13.8$ (approximately). Most people would not guess that the values of the exponential function have already surpassed a million when x is only 14! ■



URE 15

Exercises 1.5

(a) Write an equation that defines the exponential function with base $a > 0$.

(b) What is the domain of this function?

(c) If $a \neq 1$, what is the range of this function?

(d) Sketch the general shape of the graph of the exponential function for each of the following cases.

(i) $a > 1$ **(ii)** $0 < a < 1$

(a) How is the number e defined?

(b) What is an approximate value for e?

(c) What is the natural exponential function?

3–6 ■ Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

4. $y = e^x, \quad y = e^{-x}, \quad y = 8^x, \quad y = 8^{-x}$

5. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x, \quad y = 0.6^x, \quad y = 0.3^x, \quad y = 0.1^x$

7–12 ■ Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 2 and

13 and, if necessary, the transformations of Section 1.2. Indicate the location of the horizontal asymptote.

7. $y = 4^x - 3$

8. $y = 4^{x-3}$

9. $y = -2^{-x}$

10. $y = 2e^x + 1$

11. $f(x) = 3e^{-x}$

12. $g(x) = 2\left(\frac{1}{2}\right)^x - 1$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from

(a) shifting 2 units downward

(b) shifting 2 units to the right

(c) reflecting about the x-axis

(d) reflecting about the y-axis

(e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from

(a) reflecting about the x-axis and then shifting 4 units to the left

(b) reflecting about the y-axis and then shifting 3 units upward

15–20 ■ Simplify each of the following expressions.

15. x^3x^5

16. b^9/b^3

17. $(u^4)^2$

18. $(m^2n)^4$

19. $\left(\frac{p^3}{2}\right)^3$

20. $(2xy^2)^3$

21–24 ■ Write each of the following as an expression using radicals.

21. $4^{2/3}$

22. $7^{5/2}$

23. $e^{1/4}$

24. $w^{3/4}$

25–30 ■ Show that each of the following statements is true.

25. $P \cdot 3^{3x} = P \cdot 27^x$

26. $8^{t/3} = 2^t$

27. $500 \cdot (1.025)^{4t} \approx 500 \cdot (1.1038)^t$

28. $\frac{1}{e^{x/2}} = \left(\frac{1}{\sqrt{e}}\right)^x$

29. $4^{x+3} = 64 \cdot 4^x$

30. $12e^{0.2t} \approx 12 \cdot (1.2214)^t$

31–34 ■ The table lists some function values. Decide whether the function could be linear, exponential, or neither. If the function could be linear or exponential, write a possible equation for the function.

31.

x	$f(x)$
0	5
1	10
2	20
3	40
4	80

32.

x	$g(x)$
0	5
1	10
2	15
3	20
4	25

33.

t	$A(t)$
1	12
2	11
3	9
4	6
5	2

34.

n	$P(n)$
1	18
2	6
3	2
4	2/3
5	2/9

35. Bacteria population Under ideal conditions a certain bacteria population is known to double every three hours. Suppose that there are initially 100 bacteria.

(a) What is the size of the population after 15 hours?

(b) What is the size of the population after t hours?

(c) Estimate the size of the population after 20 hours.

(d) Graph the population function and estimate the time for the population to reach 50,000.

36. Bacteria population A bacteria culture starts with 500 bacteria and doubles in size every half hour.

(a) How many bacteria are there after 3 hours?

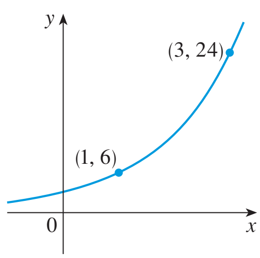
(b) How many bacteria are there after t hours?

(c) How many bacteria are there after 40 minutes?

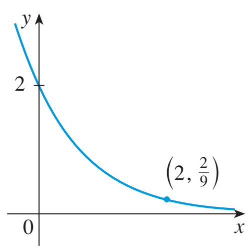
(d) Graph the population function and estimate the time for the population to reach 100,000.

37–38 ■ Find the exponential function $f(x) = C \cdot a^x$ whose graph is given.

37.



38.



39. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

40. **Compensation** Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

I. One million dollars at the end of the month.

II. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the nth day.

41. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

42. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

43. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

44. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

45. **World population** Use a graphing calculator with exponential regression capability to model the population of the world with the data from 1950 to 2010 in Table 1 on page 58. Use the model to estimate the population in 1993 and to predict the population in the year 2020.

46. **World population**

(a) Use a graphing calculator to find an exponential model for the population of the world with the data from 1900 to 1950 in Table 1 on page 58.

(b) Use your results from part (a) and Exercise 45 to write a piecewise function that models the world population for 1900 to 2010.

47. **Computing power** *Moore's Law*, named after Gordon Moore, the co-founder of Intel Corporation, is an observation that computing power increases exponentially. One formulation of Moore's Law states that the number of transistors on integrated circuits doubles every 18 months. The table lists the number of transis-

Year	Processor	Transistors (in millions)
1982	80286	0.134
1985	386	0.275
1989	486	1.2
1993	Pentium	3.1
1995	Pentium Pro	5.5
1997	Pentium II	7.5
1999	Pentium III	28
2001	Pentium 4	42

(a) Use a graphing calculator to find an exponential model for these data. (Use $t = 0$ to represent 1980.)

(b) Use the model to estimate how long it takes for the number of transistors to double. How close is this to Moore's Law?

(c) In 2004, Intel introduced the Itanium 2 processor carrying 592 million transistors. How does this compare with the number predicted by the model in part (a)?

48. **US population** The table gives the population of the United States, in millions, for the years 1900–2010.

Year	Population	Year	Population
1900	76	1960	181
1910	92	1970	205
1920	106	1980	227
1930	123	1990	249
1940	132	2000	281
1950	152	2010	309

Use a graphing calculator with exponential regression capability to model the US population since 1900. Use the model to estimate the population in 1925 and to predict the population in the year 2020.

49. **Animal population** Some populations at first increase with exponential growth but eventually slow down and stabilize at a particular level, called the *carrying capacity*. Such quantities can be modeled by functions of the form

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

called *logistic functions*. The value M is the carrying capacity. Suppose an animal population, in thousands, is modeled by

$$P(t) = \frac{23.7}{1 + 4.8e^{-0.2t}}$$

where t is the number of years after January 1, 2000.

(a) According to the model, what is the animal popula-

(b) What is the carrying capacity of the population?

(c) Sketch a graph of the function. Then use the graph to estimate when the number of animals reaches half the carrying capacity.

50. Market penetration Consumer ownership of a particular product (such as a refrigerator or microwave oven) over time can sometimes follow a logistic model; ownership increases swiftly at first, but eventually market saturation occurs and virtually everyone who is capable of and interested in owning the product has purchased it. Suppose the percentage of households owning a certain product is given by

$$g(t) = \frac{0.94}{1 + 2.5e^{-0.3t}}$$

where t is the number of years after 1980.

(a) The carrying capacity is 0.94. What does this represent in this context? Why can't the carrying capacity be larger than 1?

(b) What percentage of households owned the product in 1990?

(c) Use a graphing calculator to estimate when 90% of households owned the product.

Challenge Yourself

51. Starting with the graph of $y = 2^x$, find the equation of the graph that results from

(a) reflecting about the line $y = 3$

(b) reflecting about the line $x = -4$

52. Starting with the graph of $y = e^x$, find the equation of the graph that results from

(a) reflecting about the line $y = 4$

(b) reflecting about the line $x = 2$

53. If you graph the function $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$ you'll see that f appears to be an odd function. Prove it.

54. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

1.6 Logarithmic Functions

Introduction to Logarithms

In Section 1.5 we looked at a bacteria population that started with 1000 bacteria and doubled every hour. If t is the time in hours and N is the population in thousands, then we can say N is a function of t: $N = f(t)$. Several values are listed in Table 1. Suppose, however, that we change our point of view and become interested in the time required for the population to reach various levels. In other words, we are thinking of the function in reverse: We would like to input the population N and receive the number of hours t as the output, so $t = g(N)$. This function g is called the *inverse function* of f. Its values are shown in Table 2; they are simply the

TABLE 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t (in thousands)
0	1
1	2
2	4
3	8
4	16
5	32

TABLE 2 t as a function of N

N	$t = g(N)$ = time (in hours) to reach N thousand bacteria
1	0
2	1
4	2
8	3
16	4
32	5

values from Table 1 with the columns reversed. The inputs of f become the outputs of g, and vice versa.

The inverse of the exponential function $f(x) = a^x$ (assuming that $a > 0$ and $a \neq 1$) is called the **logarithmic function with base *a*** and is denoted by $\log_a$. The population of the bacteria in Table 1 is given by $f(t) = 2^t$, so its inverse (in Table 2) is $g(N) = \log_2 N$. In words, the value of $\log_2 N$ is the exponent to which the base 2 must be raised to give N. Since $f(3) = 2^3 = 8$, we have $g(8) = \log_2 8 = 3$. In general,

$$\log_a b = c \iff a^c = b$$

■ EXAMPLE 1 Evaluating a Logarithm

The value of $\log_5 125$ is 3, because $5^3 = 125$. ■

■ EXAMPLE 2 Converting between Logarithmic and Exponential Forms

Write the logarithmic expression $\log_4 w = r$ in an equivalent exponential form.

SOLUTION

In the logarithmic expression, r is the exponent to which 4 is raised to get w:

$$4^r = w$$ ■

The most commonly used bases for logarithms are 10 and e. In fact, these are normally the only bases for which calculators have logarithm keys. When the base is 10, the subscript 10 is often omitted. Thus $\log x$ is assumed to be the logarithmic function with base 10, called the **common logarithm**. It is the inverse of the exponential function $y = 10^x$.

The Natural Logarithmic Function

For the purposes of calculus, we will soon see that the most convenient choice of a base for logarithms is the number e, which was defined in Section 1.5. The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

ation for Logarithms

t textbooks in calculus and the nces, as well as calculators, use notation $\ln x$ for the natural logan and $\log x$ for the common rithm. In the more advanced nematical and scientific literaand in computer languages, ever, the notation $\log x$ often otes the natural logarithm.

The natural logarithmic function is the inverse of the natural exponential function e^x. Thus

(1) $$\ln b = c \iff e^c = b$$

In particular,

$$\ln e = 1$$

because the exponent to which e must be raised to return e is 1, and

$$\ln 1 = 0$$

because $e^0 = 1$.

The natural logarithmic function $\ln x$ has domain $(0, \infty)$ and range $\mathbb{R}$. (Because $\ln x$ is the inverse of e^x, its domain is the range of e^x, and its range is the domain of e^x.) The graph of $y = \ln x$, shown in Figure 1, is the reflection of the graph of $y = e^x$ about the line $y = x$. The logarithmic function has a vertical asymptote along the y-axis and x-intercept 1, whereas the exponential function has a horizontal asymptote along the x-axis and y-intercept 1. The fact that $y = e^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \ln x$ is a very slowly increasing function for $x > 1$. Notice that the values of $\ln x$ become very large negative as x approaches 0.

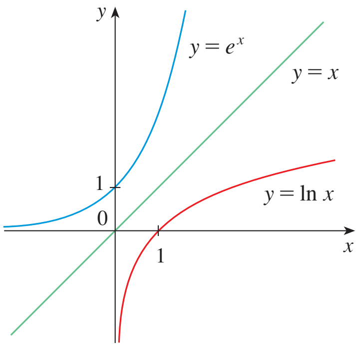


FIGURE 1

■ EXAMPLE 3 Sketching the Graph of a Logarithmic Function

Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION

We start with the graph of $y = \ln x$ as given in Figure 1. Using the transformations of Section 1.2, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ and then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure 2.)

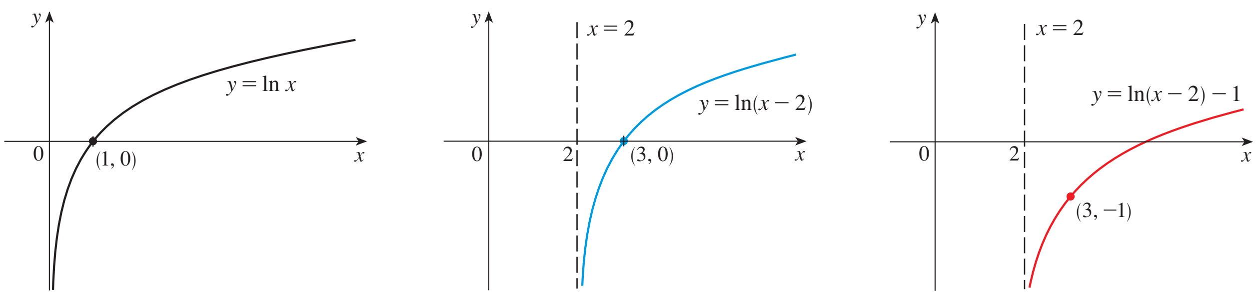


FIGURE 2

■

Although $\ln x$ is an increasing function, it grows *very* slowly when $x > 1$. In fact, $\ln x$ grows more slowly than any positive power of x. (Compare this to the fact that e^x grows more rapidly than any power of x.) To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 3 and 4. You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316

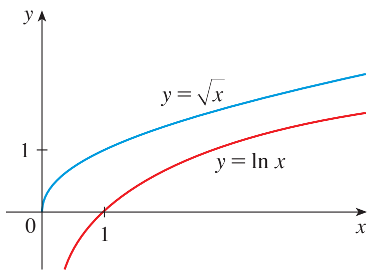


FIGURE 3

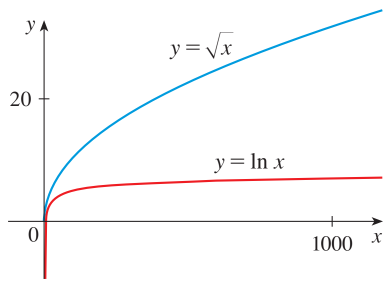


FIGURE 4

Properties of Logarithms

The following **cancellation equations** say that if we form the composition of the natural logarithmic and exponential functions, in either order, the output is simply the original input.

(2)

$$\ln(e^x) = x$$
$$e^{\ln x} = x \qquad (x > 0)$$

Because the exponential function and logarithmic function are inverses of each other, these equations say in effect that the two functions cancel each other when applied in succession.

■ EXAMPLE 4 Evaluating Natural Logarithms

Evaluate:

(a) $\ln(e^4)$ **(b)** $\ln 25$

SOLUTION

(a) From the first cancellation equation in (2), $\ln(e^4) = 4$. Another way to look at it: $\ln(e^4) = 4$ is the exponent to which e must be raised to get e^4, namely 4.

(b) The value of $\ln 25$ is the exponent to which e must be raised to get 25, but this is not a number we can determine by hand. Using a calculator, the value is approximately 3.2189. Thus $e^{3.2189} \approx 25$. ■

■ EXAMPLE 5 Solving a Basic Logarithmic Equation

Find x if $\ln x = 5$.

SOLUTION 1

From (1) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2

Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (2) says that $e^{\ln x} = x$. Therefore

$$x = e^5$$ ■

The following properties of logarithmic functions follow from the corresponding properties of exponential functions given in Section 1.5.

■ Laws of Logarithms If x and y are positive numbers, then

1. $\ln(xy) = \ln x + \ln y$

2. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$

3. $\ln(x^r) = r \ln x$ (where r is any real number)

■ EXAMPLE 6 Simplifying a Logarithmic Function

Show that $f(t) = \ln(5e^{3t})$ is a linear function.

SOLUTION

Using the first law of logarithms, $f(t)$ can be written as $\ln 5 + \ln(e^{3t})$. But the first cancellation equation in (2) says that $\ln(e^{3t}) = 3t$, so we have $f(t) = 3t + \ln 5$, a linear function with slope 3 and y-intercept $\ln 5 \approx 1.6094$. ■

■ EXAMPLE 7 Using Properties of Logarithms

Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION

Using Laws 3 and 1 of logarithms, we have

$$\begin{aligned} \ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2} \\ &= \ln a + \ln\sqrt{b} \\ &= \ln\left(a\sqrt{b}\right) \end{aligned}$$

■

Solving Exponential Equations

Logarithms can be used to solve exponential equations. By taking the natural logarithm of each side of an equation, we can use the properties of logarithms to solve for a variable in an exponent regardless of the base of the exponential expression.

■ EXAMPLE 8 Solving an Exponential Equation

Solve the equation $e^{5-3x} = 10$.

SOLUTION

We take natural logarithms of both sides of the equation and use (2):

$$\begin{aligned} \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \tfrac{1}{3}(5 - \ln 10) \end{aligned}$$

Using a calculator, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

■ EXAMPLE 9 Solving an Exponential Equation

Solve the equation $3^x = 18$. Give a decimal number solution, rounded to four decimal places.

SOLUTION

Take natural logarithms of both sides of the equation:

$$\ln(3^x) = \ln 18$$

Using Law 3 of logarithms, we can write

$$x \cdot \ln 3 = \ln 18$$

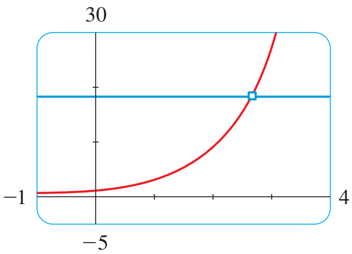


FIGURE 5

We can use a graphing calculator to check our work in Example 9. Figure 5 shows that the graph of $y = 3^x$ intersects the horizontal line $y = 18$ at $x \approx 2.63$.

Dividing both sides by ln 3 gives

$$x = \frac{\ln 18}{\ln 3} \approx 2.6309$$

A quick check confirms that $3^{2.6309} \approx 18$. ■

■ EXAMPLE 10 An Exponential Model for Light Intensity

Light decreases in intensity exponentially as it passes through a substance. Suppose the intensity of a beam of light passing through the murky water in a pond can be modeled by $I(x) = I_0 \cdot 2^{-x/18}$, where I_0 is the initial intensity of the light and x is the distance in feet that the beam has traveled through the water. How far has the beam traveled when its intensity is reduced to 10% of its original intensity?

SOLUTION

Because 10% of the original intensity is $0.10I_0$, we need to solve the equation $I_0 \cdot 2^{-x/18} = 0.1I_0$. We start by isolating the exponential expression (divide both sides by I_0), and then we take natural logarithms of both sides:

$$2^{-x/18} = 0.1$$

$$\ln(2^{-x/18}) = \ln(0.1)$$

$$-\frac{x}{18} \cdot \ln 2 = \ln(0.1)$$

$$x = -\frac{18}{\ln 2} \cdot \ln(0.1) \approx 59.795$$

Thus the intensity is reduced to 10% of the original intensity after the light has passed through about 59.8 feet of the water. ■

■ Exercises 1.6

1. **(a)** How is the logarithmic function $y = \log_a x$ defined?
   (b) What is the domain of this function?
   (c) What is the range of this function?

2. **(a)** What is the natural logarithm?
   (b) What is the common logarithm?
   (c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–6 ■ Find the exact value (without using a calculator) of each expression.

3. **(a)** $\log_2 64$ **(b)** $\log_6 \frac{1}{36}$

4. **(a)** $\log_2 8$ **(b)** $\log_8 2$

5. **(a)** $\ln e^3$ **(b)** $e^{\ln 7}$

6. **(a)** $\ln e^{\sqrt{2}}$ **(b)** $e^{3 \ln 2}$

7–10 ■ Use a calculator to evaluate the quantity correct to four decimal places.

7. $\ln 100$ **8.** $3 \ln(e + 2)$

9. $\dfrac{\ln 28}{\ln 4}$ **10.** $\dfrac{\ln 6}{5 \ln 3}$

11–12 ■ Write the logarithmic expression in an equivalent exponential form.

11. **(a)** $\log_8 4 = \frac{2}{3}$ **(b)** $\log_6 u = v$

12. **(a)** $\ln 12 \approx 2.4849$ **(b)** $C = \ln A$

13–14 ■ Write the exponential expression in an equivalent logarithmic form.

13. **(a)** $10^3 = 1000$ **(b)** $y = 4^x$

14. **(a)** $e^x = 2$ **(b)** $R = e^{3t}$

15–18 ■ Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 1 and the transformations of Section 1.2.

15. $y = -\ln x$ **16.** $y = \ln(-x)$

17. $y = \ln(x + 1) + 3$ **18.** $y = \ln(x - 4) - 2$

---

19. Starting with the graph of $y = \ln x$, find the equation of the graph that results from
(a) shifting 3 units upward
(b) shifting 3 units to the left
(c) reflecting about the x-axis
(d) reflecting about the y-axis

20. If we start with the graph of $y = \ln x$, reflect the graph about the x-axis, and then shift the graph down 4 units, what is the equation of the resulting graph?

21. Suppose that the graph of $y = \ln x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

22. Compare the functions $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?

23–26 ■ State whether each of the following is true or false.

23. $\ln(c + d) = \ln c + \ln d$ **24.** $\ln(cd) = \ln c + \ln d$

25. $\ln(u/3) = \dfrac{\ln u}{\ln 3}$ **26.** $(\ln x)^2 = 2 \ln x$

27–30 ■ Express the given quantity as a single logarithm.

27. $2 \ln 4 - \ln 2$ **28.** $\ln 3 + 2 \ln x$

29. $3 \ln u - 2 \ln 5$ **30.** $\ln x + a \ln y - b \ln z$

---

31. **(a)** Explain why $y = \ln(x^3)$ and $y = 3 \ln x$ have the same graph.
(b) Explain why $y = \ln(x^2)$ and $y = 2 \ln x$ don't have the same graph.

32. The graph of the function $f(t) = \ln(3^t)$ is a line through the origin. Explain why this is a linear function. What is the slope?

33–34 ■ Solve each equation for x. Give both an exact solution and a decimal approximation, rounded to four decimal places.

33. **(a)** $2 \ln x = 1$ **(b)** $e^{-x} = 5$

34. **(a)** $e^{2x+3} - 7 = 0$ **(b)** $\ln(5 - 2x) = -3$

35–42 ■ Solve each equation. Give a decimal approximation, rounded to four decimal places.

35. $5^t = 20$ **36.** $1.13^x = 7.65$

37. $2^{x-5} = 3$ **38.** $10^{3-2x} = 42$

39. $8e^{3x} = 31$ **40.** $450e^{0.15t} = 1200$

41. $6 \cdot (2^{x/7}) = 11.4$ **42.** $100 \cdot (4^{-p/5}) = 8.8$

---

43. County population Suppose the function $P(t) = 437.2(1.036)^t$ is used to model the population, measured in thousands of people, of a county t years after the end of 1995. When will the population reach one million people?

44. Vehicle value The value of Tracy's car is given by $V(t) = 18500(0.78)^t$, where t is the number of years she has owned the vehicle. When will the car be worth only \$2000?

45. Water transparency Environmental scientists measure the intensity of light at various depths in a lake to find the "transparency" of the water. Certain levels of transparency are required for the biodiversity of the submerged macrophyte population. In a certain lake the intensity of light at a depth of x feet is given by

$$I = 10e^{-0.008x}$$

where I is measured in lumens. At what depth has the light intensity dropped to 5 lumens?

46. Engine temperature Suppose you're driving a car on a cold winter day (20°F outside) and the engine overheats (at about 220°F). When you park, the engine begins to cool down. The temperature T of the engine x minutes after you park satisfies the equation

$$\ln\left(\frac{T - 20}{200}\right) = -0.11x$$

Find the temperature of the engine after 20 minutes.

47. Bacteria population If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is

$$n = f(t) = 100 \cdot 2^{t/3}$$

(see Exercise 35 in Section 1.5). When will the population reach 50,000?

48. Electric charge When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

49. Investment Many graphing calculators can fit a logarithmic function $f(x) = a + b \ln x$ to data. The table shows the time required for a $10,000 investment to reach different values in a particular bank account.

Value	Years
$11,000	2.1
$12,000	4.0
$13,000	5.8
$14,000	7.4
$15,000	9.0
$16,000	10.4
$17,000	11.7
$18,000	13.0

(a) Use a graphing calculator to find a logarithmic model for the data.

(b) Use the model to estimate how long it will take for the account to reach $25,000 in value.

50. Kiln temperature A pottery kiln heated to 2400°F is turned off and allowed to cool. An alert sounds whenever the temperature drops 200°F . The elapsed times, in hours, when the alerts sounded are recorded in the table.

Temperature (°F)	Time (hours)
2200	0.52
2000	1.12
1800	1.80
1600	2.56
1400	3.45
1200	4.48
1000	5.76
800	7.39

(a) Use a graphing calculator to find a logarithmic model for the data.

(b) Use the model to estimate how long it will take for the kiln to cool to 300°F.

Challenge Yourself

51. Television viewership Market researchers estimate that the percentage of households that have viewed a particular television program is given by the logistic function

$$f(t) = \frac{0.41}{1 + 0.52e^{-0.4t}}$$

where t is time in years and $t = 0$ corresponds to January 1, 2005. When will 30% of households have seen the program?